XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 8 form
Ratmino, July 31, 2022

1. (I.Kukharchuk) Let $ABCD$ be a convex quadrilateral with $\angle BAD = 2\angle BCD$ and $AB = AD$. Let P be a point such that $ABCP$ is a parallelogram. Prove that $CP = DP$.

   Solution. We obtain that A is the reflection about BD of the circumcenter O of triangle BCD, i.e. $ABOD$ is a rhombus. Then the segment OD is equal and parallel to AB, and therefore to CP. Hence $CODP$ is a parallelogram, and since $OC = OD$ this parallelogram is a rhombus, i.e. $CP = DP$ (fig. 8.1).

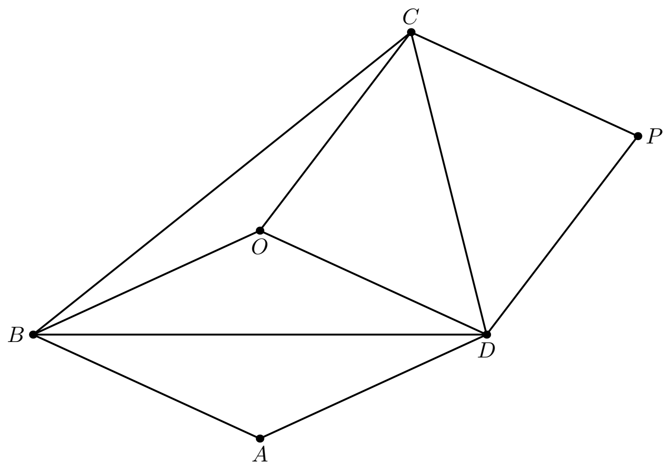

Fig. 8.1.

2. (A.Mardanov) Let $ABCD$ be a right-angled trapezoid and M be the midpoint of its greater lateral side CD. Circumcircles ω_1 and ω_2 of triangles BCM and AMD meet for the second time at point E. Let ED meet ω_1 at point F, and FB meet AD at point G. Prove that GM bisects angle BGD.

   Solution. Since $\angle BEM = 180^\circ - \angle C = \angle D$, we obtain that E lies on the sideline AB. Thus $\angle CFD = 90^\circ$ and $CM = FM = MD$. Also G, F, M, D are concyclic because $\angle BGA = \angle GBC = \angle FMD$ (fig. 8.2). Hence GM bisects angle BGD because $FM = MD$.

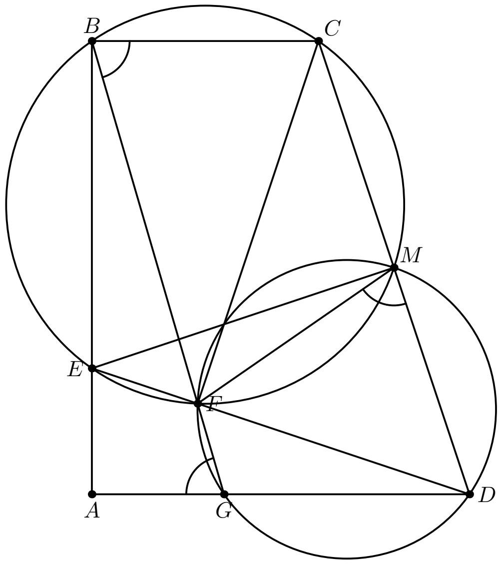

Fig. 8.2.

3. (D.Reznik, A.Zaslavsky) A circle ω and a point P not lying on it are given. Let ABC be an arbitrary regular triangle inscribed into ω and A', B', C' be the projections of P to BC, CA, AB. Find the locus of centroids of triangles $A'B'C'$.

   Answer. The midpoint of OP, where O is the center of the given circle.

   Solution. Construct the lines a, b, c, passing through and parallel to BC, CA, AB resepctively. Let PA'', PB'', PC'' be the perpendiculars from P to a, b, c. Note that A'', B'', C'' lie on the circle with diameter OP and $\angle A''B''C'' = \angle A''PC'' = 60^\circ$ (fig. 8.3). Therefore $A''B''C''$ is a regular triangle and its centroid coincide with the midpoint of OP. The centroid of $A'B'C'$ also coincide with this point because $\overrightarrow{A'A''} + \overrightarrow{B'B''} + \overrightarrow{C'C''} = \overrightarrow{0}$.

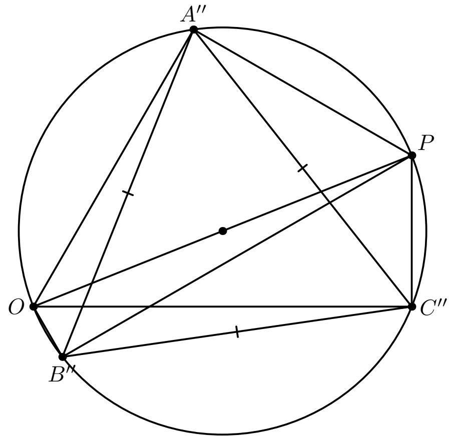

Fig. 8.3.

4. (A.Mardanov) Let $ABCD$ be a cyclic quadrilateral, O be its circumcenter, P be a common points of its diagonals, and M, N be the midpoints of AB and CD respectively. The circle OPM meets for the second time segments AP and BP at points A_1 and B_1 respectively, and the circle OPN meets for the second time segments CP and DP at points C_1 and D_1 respectively. Prove that the areas of quadrilaterals AA_1B_1B and CC_1D_1D are equal.

Solution. Since PM, PN are the medians of similar triangles PAB and PDC, and OM, ON are the perpendicular bisectors to the corresponding sides of these triangles, we have $\angle PMO = \angle PNO$, thus the radii of two circles are equal. Then $\angle OA_1C_1 = \angle OC_1A_1$, therefore $OA_1 = OC_1$ and $AA_1 = CC_1$. Similarly we obtain that $OB_1 = OD_1$ and $BB_1 = DD_1$. Let the line passing through P and perpendicular to OP meet AB and CD at points M_1, N_1 respectively. Since $\angle OMM_1 = \angle ONN_1 = 90°$, these points on the circles OMP and ONP respectively, and $OM_1 = ON_1$. Then the triangles OM_1A_1 and ON_1C_1 are congruent by two sides and an angle, i.e. $A_1M_1 = C_1N_1$. Similarly $B_1M_1 = D_1N_1$ and $A_1B_1 = C_1D_1$. Thus the triangles $A_1B_1M_1$ and $C_1D_1N_1$ are congruent. Also the altitudes of triangles M_1BB_1 and N_1DD_1 from M_1 and N_1 are equal because they are symmetric with respect to P, which yields that the areas of these triangles are equal (fig. 8.4). Similarly the areas of triangles M_1AA_1 and N_1CC_1 are equal. From this we obtain the required equality of areas.

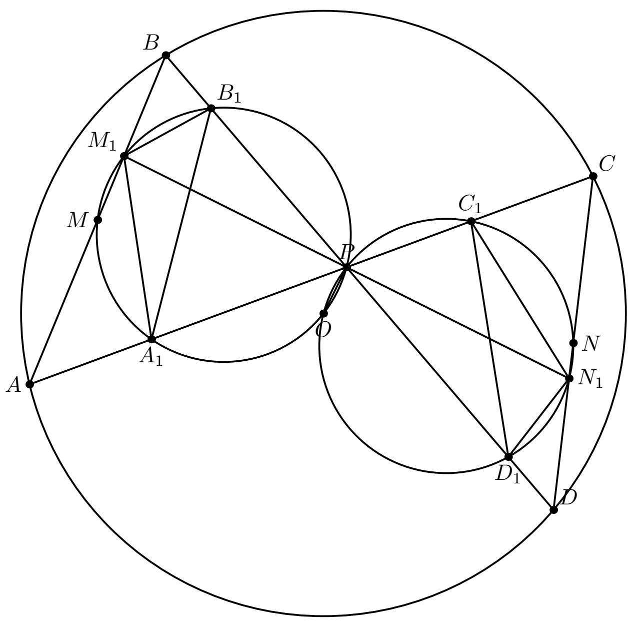

Fig. 8.4.

Remark. We can also obtain that $OM_1 = ON_1$ from the butterfly theorem and obtain from this all remaining equalities.

XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8 form

Ratmino, August 1, 2022

5. (D.Shvetsov) The incircle of triangle ABC touches AB, BC, AC at points C_1, A_1, B_1 respectively. Let A' be the reflection of A_1 about B_1C_1; point C' is defined similarly. Lines $A'C_1$ and $C'A_1$ meet at point D. Prove that $BD \parallel AC$.

   Solution. We have $\angle A'C_1B_1 = \angle A_1C_1B_1 = (180^\circ - \angle C)/2$, therefore $\angle DC_1A_1 = \angle C$. Similarly we obtain that $\angle DA_1C_1 = \angle A$. Then $\angle C_1DA_1 = \angle B$. Thus A_1BDC_1 is a cyclic quadrilateral and $\angle DBA = \angle DA_1C_1 = \angle BAC$, which yields the required assumption (fig. 8.5).

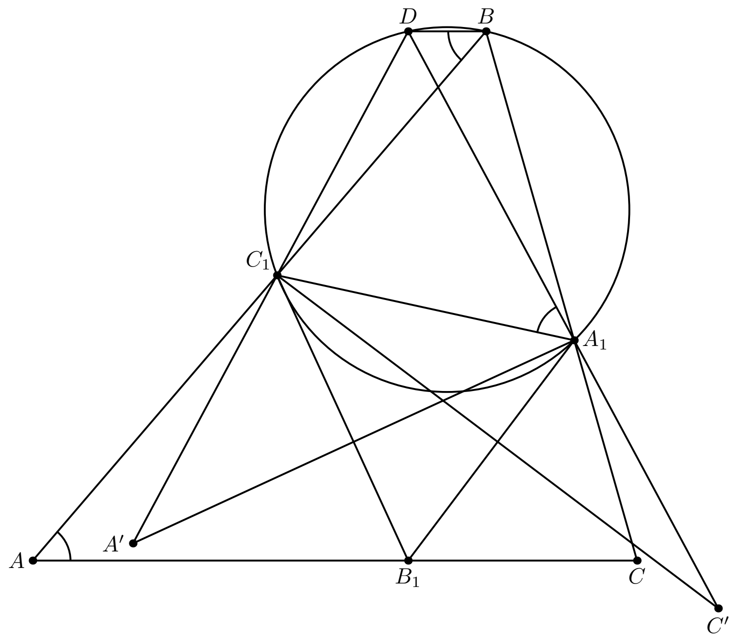

Fig. 8.5.

6. ($\boxed{\text{A.Bremzen}}$, A.Kulakov) Two circles meeting at points A, B and a point O outside them are given. Using a compass and a ruler, construct a ray with origin O meeting the first circle at point C and the second one at point D in such a way that the ratio $OC : OD$ be maximal.

   Solution. Consider a homothety with center O and coefficient $OC : OD$. It maps the second circle ω_2 to some circle ω passing through C. If the ratio $OC : OD$ is maximal, then any circle homothetic to ω with center O and coefficient greater than 1 does not intersect the first circle ω_1. Therefore ω is tangent to ω_1 at C, i.e. the tangents to ω_1 and ω_2 at C and D respectively are parallel, and CD passes through the center I of the internal homothety of these circles. From this we obtain the required construction: C is the farest from O common point of ω_1 and the line OI, and D is the nearest to O common point of OI and ω_2 (fig. 8.5).

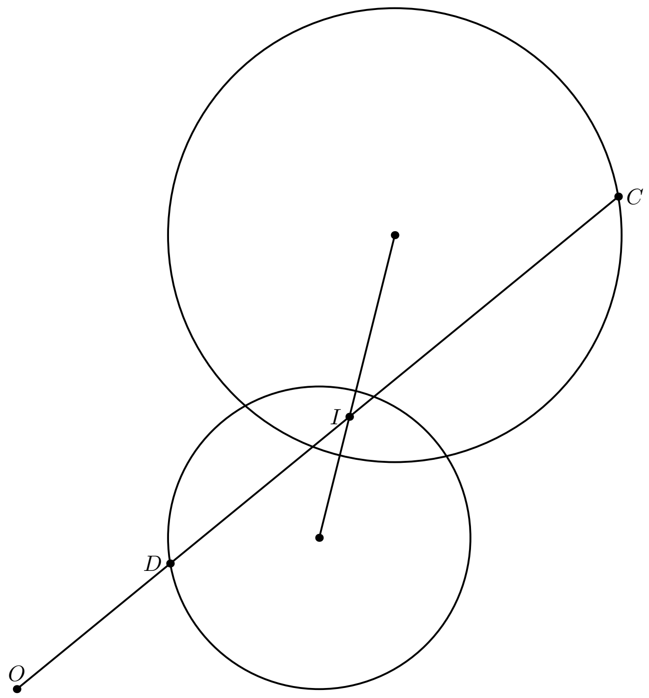

Fig. 8.6.

7. (A.Shapovalov) Ten points on a plane are such that any four of them lie on the boundary of some square. Is it obligatory true that all ten points lie on the boundary of some square?

   Answer. No.

   Solution. Prove that the vertices of a cyclic quadrilateral lie on the boundary of some square. If $ABCD$ is cyclic then there are two adjacent non-acute angles, let they are angles A and B. Thus the projections X, Y of C, D respectively to AB lie outside the segment AB. Let $CX \leq DY$, then the vertices of the quadrilateral lie on the boundary of rectangle $XYDZ$, where Z is the projection of D to CX (fig. 8.7). Now, if $DY > DZ$, then extend the segments XY and ZD beyond Y and D respectively, and if $DY < DZ$, then extend YD and XZ beyond D and Z.

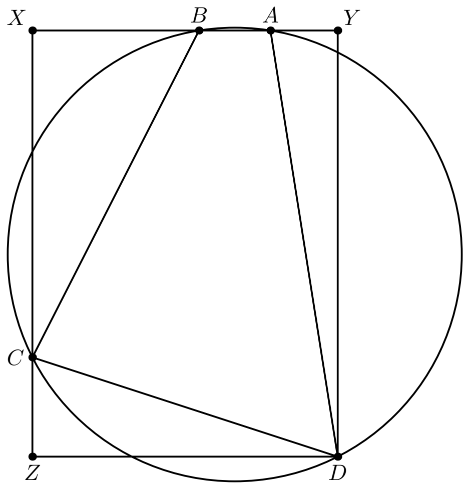

Fig. 8.7.

Now consider a cyclic decagon. Its vertices can not lie on the boundary of any square, because such boundary has at most eight common points with a circle. But as proved above any four vertices lie on the boundary of some square.

8. (I.Kukharchuk) An isosceles trapezoid $ABCD$ ($AB = CD$) is given. A point P on its circumcircle is such that segments CP and AD meet at point Q. Let L be the midpoint of QD. Prove that the diagonal of the trapezoid is not greater than the sum of distances from the midpoints of the lateral sides to an arbitrary point of line PL.

   Solution. Let E be the midpoint of AB, F be the midpoint of CD, G be the midpoint of CQ, and E_1 be the reflection of E about PL. Prove that $E_1F = AC$ (this is sufficient by a known lemma). For this prove that the triangles LE_1F and AGC are congruent. In fact, $AG = EL = E_1L$ (the first equality from an isosceles trapezoid $AEGL$, the second one from the symmetry), $LF = QC/2 = GC$, $\angle PLE_1 = \angle ELP$. Also the pentagon $PAEGL$ is cyclic because the trapezoid $AEGL$ is isosceles and $\angle APC = \angle ADC = \angle ALG$ (fig. 8.8). Thus $\angle ELP = \angle EGP = \angle ALR$ (where R lies on the extension of FL beyond L) and $\angle RLE_1 = \angle ALE_1 - \angle ALR = \angle ALE_1 - \angle PLE_1 = \angle ALP = \angle AGP$. Hence $\angle E_1LF = \angle AGC$ and the triangles are congruent.

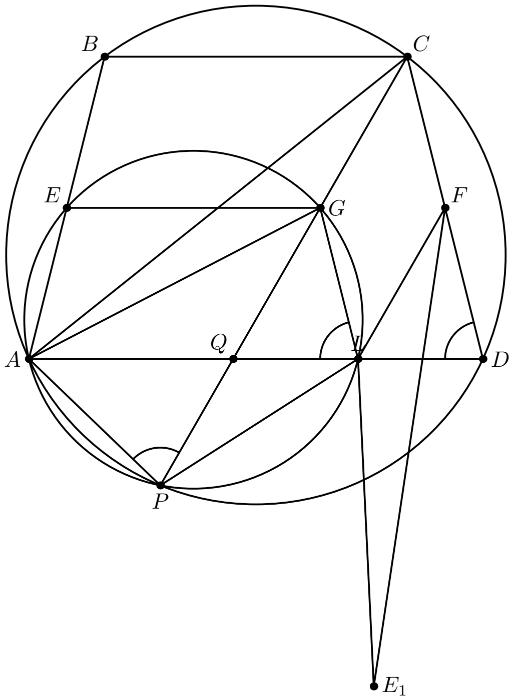

Fig. 8.8.

XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 9 form

Ratmino, July 31, 2022

1. (D.Shvetsov) Let BH be an altitude of right-angled triangle ABC ($\angle B = 90^\circ$). An excircle of triangle ABH opposite to B touches AB at point A_1; a point C_1 is defined similarly. Prove that $AC \parallel A_1C_1$.

   Solution. The segments BA_1, BC_1 are equal to the semiperimeters of triangles ABH, BCH respectively. Since these triangles are similar, we have $BA_1 : BC_1 = BA : BC$, which yields the required assumption.

2. (L.Emelyanov) Let circles s_1 and s_2 meet at points A and B. Consider all lines passing through A and meeting the circles for the second time at points P_1 and P_2 respectively. Construct by a compass and a ruler a line such that $P_1A \cdot AP_2$ is maximal.

   First solution. Let X, Y be the projections of A to BP_1, BP_2 respectively. Since angles AP_1B, AP_2B do not depend on the choice of the line, we obtain that the products $AP_1 \cdot AP_2$ and $AX \cdot AY$ obtain their maximal values simultaneously. Since X, Y lie on the circle with diameter AB, we obtain that he angle XAY, and the length of XY are constant, therefore we have to find such chord XY, that the distance from A to it is maximal. Since all chords XY touch a fixed circle centered at the midpoint of AB (fig. 9.2), we obtain the maximal distance, when the the distance from the touching point to A is maximal. Then AB bisects the angle AXY, and therefore the angle P_1BP_2. The construction of this chord is clear.

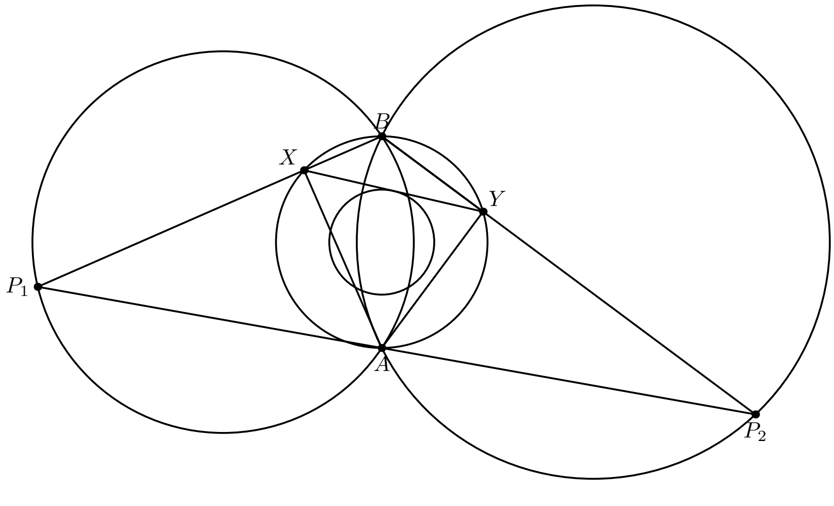

Fig. 9.2.

Since the endpoints of constructed chord may be denoted as X and Y by two ways, we obtain two possible dispositions of line P_1P_2. The line BA is the internal bisector of angle P_1BP_2 for one of them, and the external bisector for the remaining one. If the angle P_1BP_2 is obtuse, we obtain the maximal value of $AP_1 \cdot AP_2$ in the first case, if this angle is acute the maximal value is obtained in the second case. If $\angle P_1BP_2 = 90^\circ$ both values are equal.

Second solution. Applying an inversion centered at A we obtain the following problem:

A point A and lines ℓ_1, ℓ_2 are given. Construct a line passing through A and meeting ℓ_1 and ℓ_2 at points P_1 and P_2 such that $P_1A \cdot AP_2$ is minimal.

Fix ℓ_1 and apply to ℓ_2 a homothety with center A such that the distances from A to ℓ_1 and ℓ_2 will be equal. Then all products $P_1A \cdot AP_2$ are multiplied to a constant, hence the required line does not change. Let ℓ_1 and ℓ_2 meet at point C, and the perpendicular to CA at A meet ℓ_1 and ℓ_2 at points Q_1 and Q_2 respectively. Then CQ_1Q_2 is an isosceles triangle, and A is the midpoint of Q_1Q_2.

Prove that the required line is Q_1Q_2 or AC. Let m be a line passing through A and meeting ℓ_1 and ℓ_2 at points P_1 and P_2 respectively. It is sufficient to consider two cases.

In the first case P_1 lies on the segment CQ_1, and P_2 lies on the extension of CQ_2 beyond Q_2. Since $\angle P_1P_2Q_2 < \angle AQ_2C = \angle P_1Q_1Q_2$, we obtain that Q_1 lies inside the circle $(P_1P_2Q_2)$, thus $Q_1A \cdot AQ_2 < P_1A \cdot AP_2$.

In the second case P_1 lies on the segment CQ_1, and P_2 lies on the extension of CQ_2 beyond C. Since $\angle P_1P_2C < \angle P_1CA$, we obtain that the circle P_1CP_2 intersects the segment CA, thus $CA^2 < P_1A \cdot AP_2$.

To construct the required line we do the inversion and the homothety, choose the minimum of $Q_1A \cdot AQ_2$ and CA^2, and draw the corresponding line.

3. (A.Mardanov) A medial line parallel to the side AC of a triangle ABC meets its circumcircle at points X and Y. Let I be the incenter of triangle ABC and D be the midpoint of the arc AC not containing B. A point L lie on segment DI in such a way that $DL = BI/2$. Prove that $\angle IXL = \angle IYL$.

   Solution. Reflecting X about the bisector of angle B, we obtain a point X' lying on the ray BY. We have to prove that $ILYX'$ is a cyclic quadrilateral, i.e., $BI \cdot BL = BX \cdot BY$. Note, that L is the midpoint of BI_B, where I_B is the excenter. Thus we have to prove that $2BX \cdot BY = BI \cdot BI_B = AB \cdot BC$.

Let X'' be the common point of AC and BX. Then the triangles $X''BA$ and CBY are similar, because $\angle BX''A = \angle BXY = \angle BCY$ and $\angle XBA = \angle CBY$ (fig. 9.3), which yields the required equality.

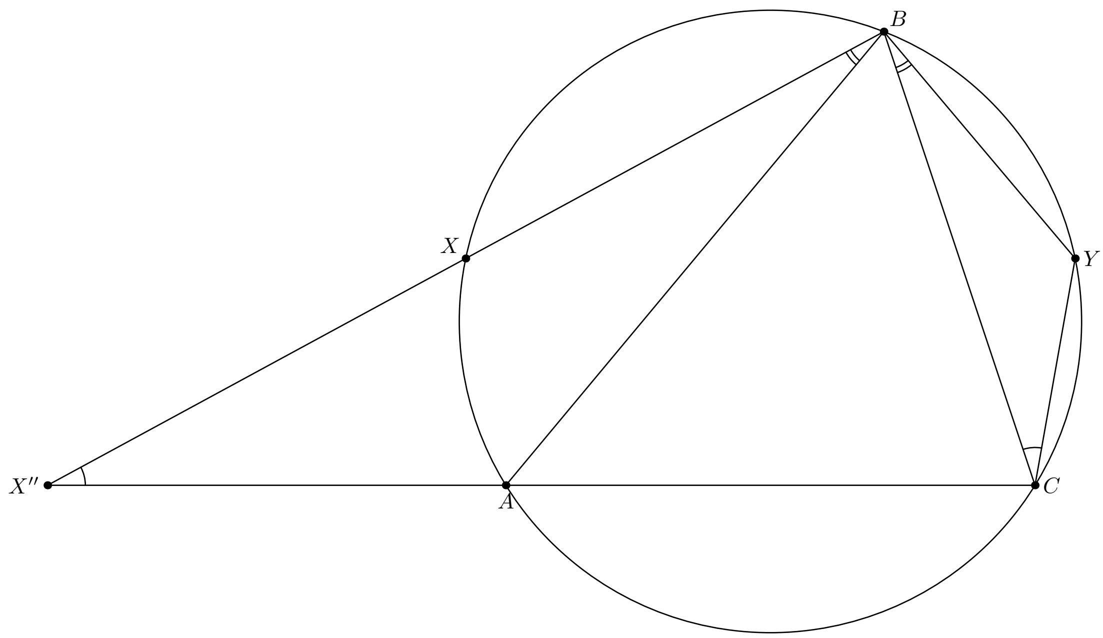

Fig. 9.3.

Remark. We can also obtain that $BI \cdot BL = BX \cdot BY$ using the composition of an inversion centered at B and the reflection about the bisector of angle ABC, swapping X and Y.

4. (B.Yakovlev) Let ABC be an isosceles triangle with $AB = AC$, P be the midpoint of the minor arc AB of its circumcircle, and Q be the midpoint of AC. A circumcircle of triangle APQ centered at O meets AB for the second time at point K. Prove that lines PO and KQ meet on the bisector of angle ABC.

   Solution. Let R, S be the midpoints of the chord AB and the minor arc AC respectively. Prove that PO and KQ meet on the circle $PRQS$.

   The spiral similarity with center P mapping the circle APQ to the circle ABC maps K to B, Q to C, and O to the circumcenter of ABC lying on PR. Therefore angle OPR equals to the angle between KQ and BC, which is equal to the angle KQR, i.e. the common point of PO and KQ lies on the circle PQR.

   Now prove that PO and the bisector BS of angle B also meet on the circle $PRQS$. Since $BS \parallel AP$ and $QS \perp AC$, we have $\angle OPQ = |90° - \angle QAP| =$

$|90^\circ - \angle CTB| = \angle BSQ$, where T is the common point of BS and AC, i.e. the quadrilateral formed by PO, PQ, QS and BS is cyclic.

So PO, KQ and BS meet the circle $PRQS$ at the same point (fig. 9.4).

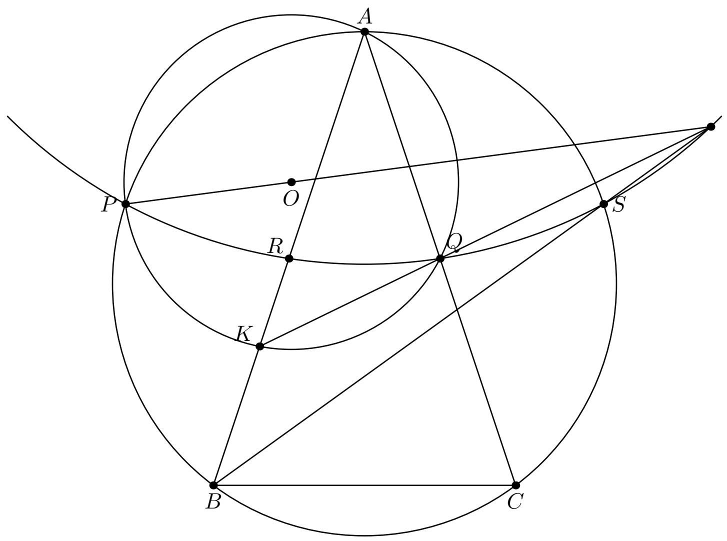

Fig. 9.4.

XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form

Ratmino, August 1, 2022

5. (A.Mardanov) Chords AB and CD of a circle ω meet at point E in such a way that $AD = AE = EB$. Let F be a point of segment CE such that $ED = CF$. The bisector of angle AFC meets an arc DAC at point P. Prove that A, E, F, and P are concyclic.

   Solution. Since AED is an isosceles triangle, we obtain that the triangle BCE is also isosceles, thus from $AD = BE$, $DF = CE = CB$, and $\angle ADF = \angle EBC$ we obtain that this triangle is congruent to the triangle AFD. Hence $PF \parallel AD$ and $\angle PFD = 180^\circ - \angle ADF = \angle AEF$, i.e. AE and PF are symmetric with respect to the perpendicular bisector to FE, which is a diameter of the circle. Therefore P and A are also symmetric with respect to this diameter and $AEFP$ is an isosceles trapezoid (fig. 9.5).

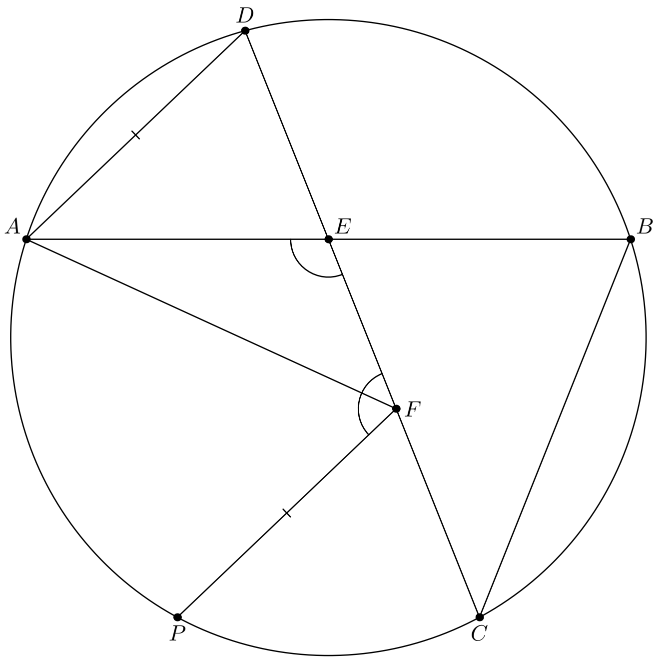

Fig. 9.5.

6. (D.Brodsky) Lateral sidelines AB and CD of a trapezoid $ABCD$ ($AD > BC$) meet at point P. Let Q be a point of segment AD such that $BQ = CQ$. Prove that the line passing through the circumcenters of triangles AQC and BQD is perpendicular to PQ.

Solution. Let the circle AQC meet for the second time AP at point X, and the circle BQD meet for the second time DP at point Y. Then $\angle AXC = \angle CQD = \angle BQA = \angle BYD$. Therefore B, C, X, Y (and thus A, D, X, Y) are concyclic (fig. 9.6), i.e. $PX : PY = PC : PB = PD : PA$ and PQ is the radical ais of circles AQC and BQD.

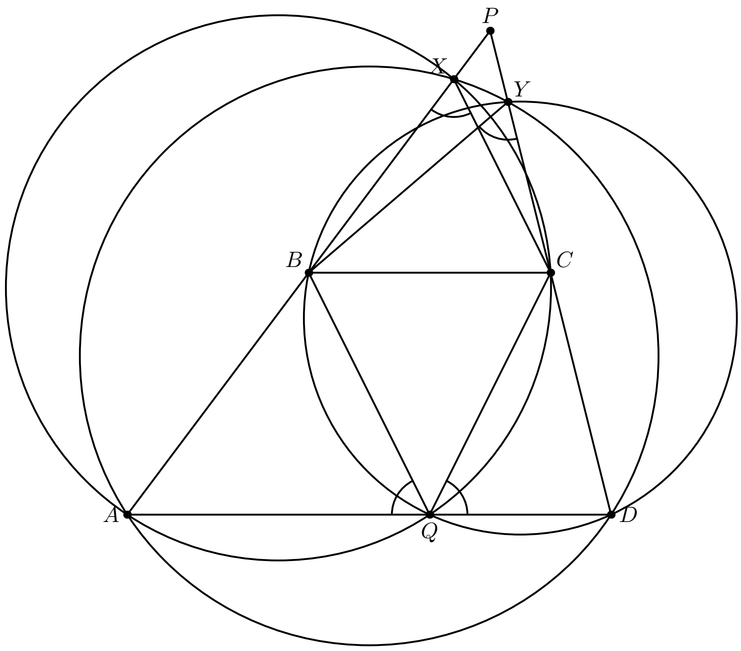

Fig. 9.6.

7. (I.Kukharchuk) Let H be the orthocenter of an acute-angled triangle ABC. The circumcircle of triangle AHC meets segments AB and BC at points P and Q. Lines PQ and AC meet at point R. A point K lies on the line PH in such a way that $\angle KAC = 90°$. Prove that KR is perpendicular to one of medians of triangle ABC.

   First solution. Since $\angle BPH = \angle ACH = \angle ABH$, we have $PH = BH$. Similarly $QH = BH$. Let L be the common point of HQ and the perpendicular to AC from C. Then $AK = KP = AP/2\sin A$ and $CL = LQ = CQ/2\sin C$. By the Menelaos theorem $AR : CR = (AP : BP)(BQ : CQ) = AK : CL$, therefore K, L, and R are collinear (fig. 9.7).

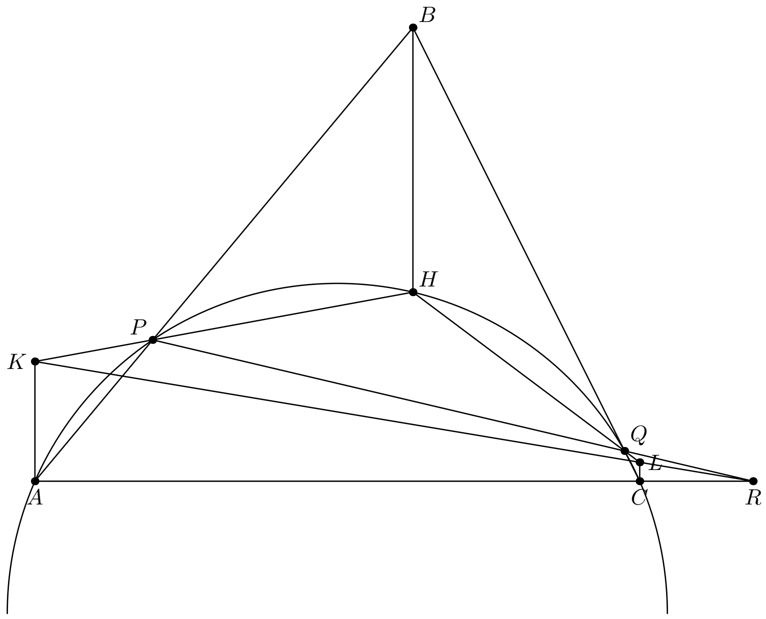

Fig. 9.7.

Now note that

$$BK^2-AK^2=\left(\frac{AB+BP}{2}\right)^2-\left(\frac{AB-BP}{2}\right)^2=AB\cdot BP=BC\cdot BQ=BL^2-CL^2,$$

Hence if M is the midpoint of AC, then $MK^2-ML^2=AK^2-CL^2=BK^2-BL^2$, i.e. $BM\perp KL$.

Second solution. We proved in the first solution that H is the circumcenter of BPQ. Thus this circle touches the circles ω_a and ω_c with centers K, L and radii KA, LC respectively. By the three homotheties theorem we obtain that R is the external homothety enter of circles ω_a and ω_c, i.e. R lies on KL.

Since AP is the common chord of circles AHC and ω_a, and CQ is the common chord of circles AHC and ω_c, we obtain that B is the radical center of these three circles. Also the degrees of M with respect to ω_a and ω_c are equal, therefore BM is the radical axis of these circles and $BM\perp KL$.

8. (F.Nilov) Several circles are drawn on the plane and all points of their intersection or touching are marked. is it possible that each circle contains exactly five marked points and each point belongs to exactly five circles?

Answer. Yes.

Solution. For each vertex of a regular icosahedron construct a circle passing through five adjacent vertices. It is clear that all such circles lie on the circumsphere of the icosahedron, Each vertex belongs to exactly five circles, and any two circles have not common points or intersect at two vertices. Hence applying a stereographic projection centered at any point not lying on these circles we obtain the required configuration.

The same example may be obtained by another way.

Mark 12 points: the vertices of regular pentagon $ABCDE$, its center O, five common points of its diagonals, and the infinite point. If P is the common point of AC and BD, then $\angle APB = 72^\circ = \angle AOB$, thus A, B, O, and P are concyclic. Draw 12 lines or circles: the diagonals of $ABCDE$, its circumcircle, the circle passing through the common points of diagonals, and the circles ABO, BCO, CDO, DEO, EAO (fig. 9.8). applying an inversion with a center not lying on these lines we obtain the required configuration.

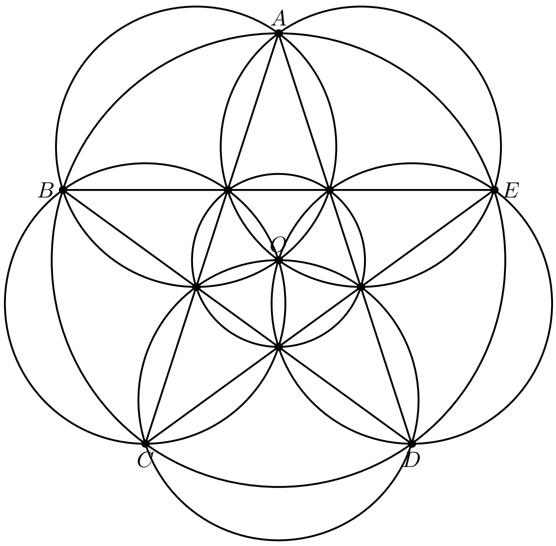

Fig. 9.8.

Remark. For any $k = 2, 3, 4, 5$ there exists a configuration of several circles and their common points, such that each circle passes through exactly k points and each point belongs to exactly k circles. It is not known does such configurations exist for $k > 5$.

XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10 form

Ratmino, July 31, 2022

1. (Tran Quang Hung) Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two squares oriented clockwisely. The perpendicular bisectors to segments A_1B_1, A_2B_2, A_3B_3, A_4B_4 meet the perpendicular bisectors to segments A_2B_2, A_3B_3, A_4B_4, A_1B_1 at points P, Q, R, S respectively. Prove that $PR \perp QS$.

   Solution. Let O be the center of a spiral similarity mapping one of the given squares to the second one, and C_i be the midpoints of A_iB_i ($i = 1, 2, 3, 4$). Then $C_1C_2C_3C_4$ is a square, and $\angle OC_1P = \angle OC_2Q = \angle OC_3R = \angle OC_4S$, i.e. OC_1PC_2, OC_2QC_3, OC_3RC_4, and OC_4SC_1 are cyclic quadrilaterals. Let the first circle meet the third one for the second time at point U, and the second circle meet the fourth one for the second time at point V. Then by the spiral similarity theorem PR passes through U, QS passes through V, and the angle between PR and QS equals to angle UOV. But it is clear that $OU \parallel C_1C_2$ and $OV \parallel C_2C_3$, thus $\angle UOV = \pi/2$.

   Remark. The assumption is also correct for two directly similar rectangles.

2. (A.Kuznetsov) Let $ABCD$ be a convex quadrilateral. The common external tangents to circles ABC and ACD meet at point E, the common external tangents to circles ABD and BCD meet at point F. Let F lie on AC, prove that E lies on BD.

   Solution. Since F is the external homothety center of circles ABD and BCD, it is also the center of an inversion mapping one of these circles to the second one. This inversion conserves the points B and D, and maps each of points A and C to the second one, therefore $AB = (BC \cdot FB)/FC$, $AD = (CD \cdot FD)/FC$, and $AB \cdot CD = AD \cdot BC$. Now let EB meet the arc ADC at point D'. Then we similarly obtain that $AD' \cdot BC = CD' \cdot AB$. The point of arc ADC with such property is unique, thus D' coincides with D, and B, D, E are collinear.

3. (G.Chelnokov) A line meets a segment AB at point C. What is the maximal number of points X of this line such that one of angles AXC and BXC is equal to a half of the second one?

   Answer. 4.

Estimation. Denote the given line (meeting the segment AB at C) as ℓ. Prove that there exists at most two points of ℓ with required property lying on the same semiplane with respect to AB.

Suppose that there are three points X_1, X_2, X_3 with required property lying on the same semiplane with respect to AB. Two cases are possible: the greatest angles in each pair lie on the same semiplane with respect to ℓ, or two greatest angles lie in the same semiplane, and the third one lies in the other semiplane. Without lost of generality let $\angle BX_1C = 2\angle AX_1C$, $\angle BX_2C = 2\angle AX_2C$.

In the first case ($\angle BX_3C = 2\angle AX_3C$) the reflection F of A about ℓ is the common point of the bisectors of tree angles, thus it lies on equal distances from ℓ and the lines X_1B, X_2B, X_3B. But the distances from F to three concurrent lines can not be equal.

In the second case ($2\angle BX_3C = \angle AX_3$) F is the common point of the bisectors of angles CX_1B and CX_2B. Prove that F and X_3 lie on the different sides with respect to AB, and therefore F can not lie on the ray symmetric to X_3A about ℓ. For this it is sufficient to prove that B and the ray CX_1 lie in the same semiplane with respect to AF.

Since $3\angle AX_1C < \pi$, $3\angle AX_2C < \pi$, we obtain that the angles AX_1C, AX_2C are acute. Hence the midpoint H of segment AF do not lie on the segment X_1X_2. Suppose that X_2 lies on the segment HX_1. Let E be the common point of rays X_2B and HF (if these rays do not intersect, the assumption is clear) (fig. 10.3).

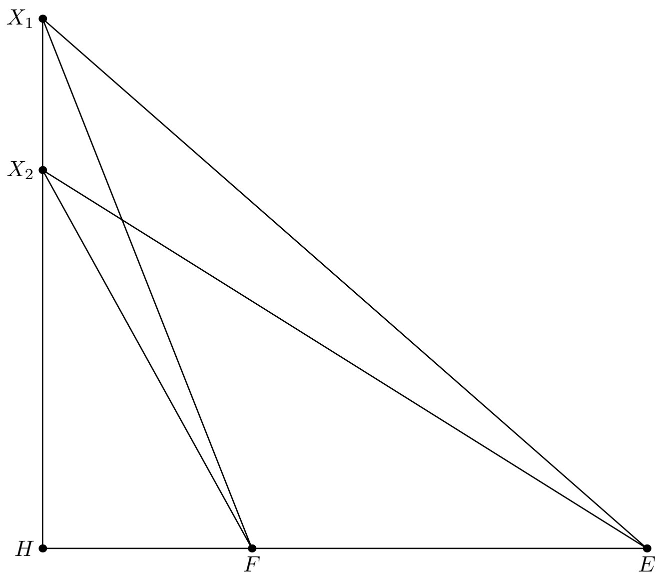

Fig. 10.3.

Prove that $\angle FX_1E > \angle FX_1H$. This yields that B, X_1, X_2 lie on the same semiplane with respect to HF, and we obtain the required assumption.

We have $HF/FE = HX_2EX_2 = \cos\angle HX_2E < \cos\angle HX_1E = HX_1/EX_1$, hence the bisector of angle HX_1E meets HE at such point P, that F lies on PH, i.e. $\angle FX_1E > \angle FX_1H$.

Example. Consider a triangle X_1AB with a median X_1C such that $\angle AX_1C = 40°$, $\angle BX_1C = 80°$. Let X_2 be such point of segment X_1C that $\angle X_1BX_2 = 20°$. Prove that $\angle X_1AX_2 = 10°$. Then X_1, X_2 and their reflections about C form the required quadruple.

Let AK, BH be the perpendicular to X_1X_2. Prove that $KA^2 = KX_1 \cdot KX_2$. Since the triangles AKC and BHC are congruent this is equivalent to the equality

$$(AX_1 \sin 40°)^2 = AX_1 \cos 40°(AX_1 \cos 40° - 2AX_1 \sin 40° \operatorname{tg} 10°).$$

It is easy to see that this is correct.

From this we obtain that the circle AX_1X_2 touches the line AK, i.e. $\angle X_2AK = \angle AX_1K = 40°$, $\angle X_1AX_2 = 10°$, $\angle AX_2C = 50°$ and $\angle BX_2C = 100°$.

4. (A.Matveev, I.Frolov) Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D$. Prove that the midpoint of BD lies on the common internal tangent to the incircles of triangles ABC and ACD.

   Solution. Let M, N C_1, A_1 be the midpoints of AC, BD, AB, BC respectively. Since $\angle A_1NC_1 = \angle D = \angle B = \angle A_1MC_1$, we obtain that N lies on the circle A_1MC_1. By the Feuerbach theorem this circle touches the incircle ω_1 of triangle ABC. Hence applying the Casey theorem to A_1, C_1, N and ω_1 we can find the length x of the tangent from N to ω_1. For ezample for the configuration of fig. 10.4 we have

   $$x\frac{AC}{2} = \frac{CD}{2} \cdot \frac{AC - BC}{2} + \frac{AD}{2} \cdot \frac{AB - AC}{2}.$$

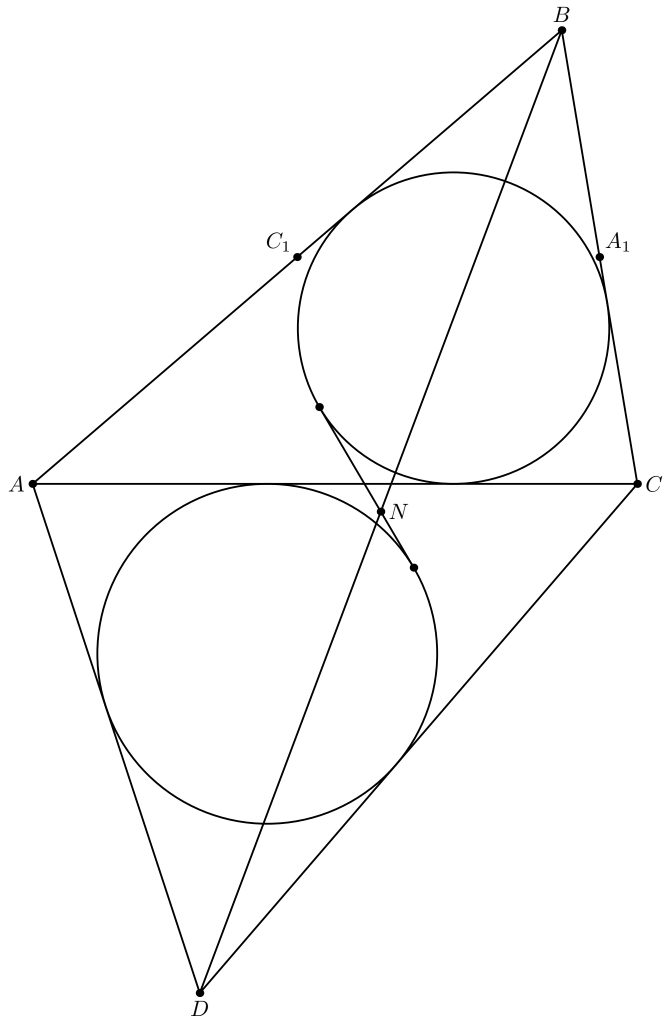

Fig. 10.4.

Similarly for the tangent y from N to the incircle ω_2 of ACD we have

$$y\frac{AC}{2} = \frac{AB}{2} \cdot \frac{CD - AC}{2} + \frac{BC}{2} \cdot \frac{AC - AD}{2}.$$

Summing these equalities we obtain that $x + y = (AB + CD - AD - BC)/2$, which equals to the length of the common internal tangent to ω_1 and ω_2, therefore N lies on such tangent. The solution for the remaining configurations is similar.

XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10 form

Ratmino, August 1, 2022

5. (A.Mardanov, K.Struikhina) Let AB and AC be the tangents from a point A to a circle Ω. Let M be the midpoint of BC and P be an arbitrary point on this segment. A line AP meets Ω at points D and E. Prove that the common external tangents to circles MDP and MPE meet on the medial line of triangle ABC.

   Solution. Let K be the midpoint of AP. Since K is the circumcenter of triangle APM, we have $KP = KM$, i.e. K lies on the line joining the centers of circles MDP and MPE. Also since A, P, D, and E form a harmonic quadruple, we have $KP^2 = KD \cdot KE$. Thus K is the external homothety center of these circles (fig.10.5).

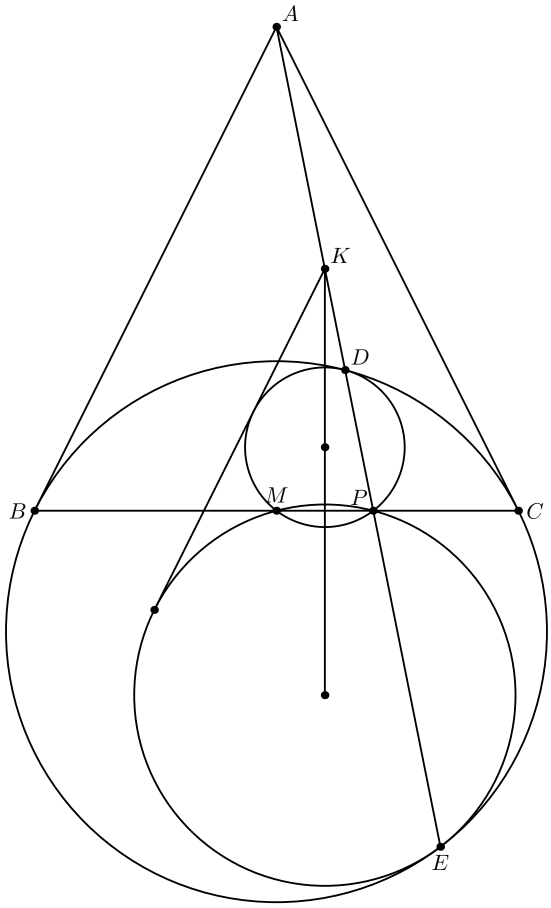

Fig. 10.5.

6. (D.Brodsky) Let O, I be the circumcenter and the incenter of triangle ABC; P be an arbitrary point on segment OI; P_A, P_B, and P_C be the second common points of lines PA, PB, and PC with the circumcircle of triangle ABC. Prove that the bisectors of angles BP_AC, CP_BA, and AP_CB concur at a point lying on OI.

   Solution. Note that for any point P the bisector of angle BP_AC meets the circumcircle for the second time at the fixed point — the midpoint of the arc BAC. Hence the common point of this bisector with OI projectively depend on P. This is also correct for the common points of OI with the bisectors of angles CP_BA и AP_CB. But when P coincides with I, all three bisectors pass through O, and when P is one of comon points of OI with the circumcircle, the bisectors meet OI at the same point. Thus for any point P all bisectors meet OI at the same point.

7. (F.Nilov) Several circles are drawn on the plane and all points of their intersection or touching are marked. May be that each circle contains exactly four marked points and each point belongs to exactly four circles?

   Answer. Yes.

   First solution. Take a square $ABCD$ with center O, its circumcircle and incircle, and four circles with diameters OA, OB, OC, OD (fig.10.7). Applying an inversion with an arbitrary center not lying on these circles and lines AB, BC, CD, DA we obtain ten circles ntersecting or touching at ten points — the images of the midpoints of the sides, the images of A, B, C, D, O, and the center of the inversion. It is easy to see that this configuration satisfies the assumption.

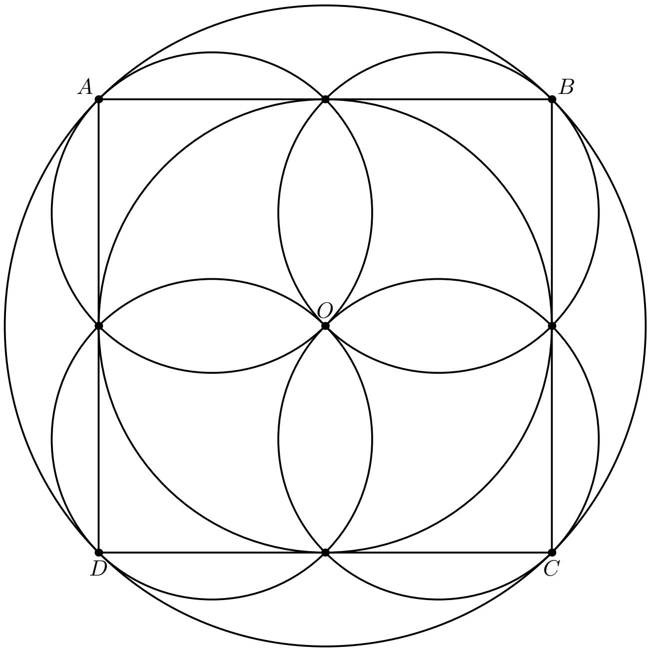

Fig. 10.7.

Second solution. Consider a cuboctahedron — the polyhedron formed by the midpoints of the edges of a cube. Construct for each its vertex the circle passing through four adjacent vertices. Clearly all these circles lie on the circumsphere of the cuboctahedron, and any two of them do not have common points, intersect at two vertices of the cuboctahedron, or touch at a vertex. Hence applying a stereographic projection from any center not lying on these circles we obtain the required configuration.

Remark. For any $k = 2, 3, 4, 5$ there exists a configuration of several circles and their common points, such that each circle passes through exactly k points and each point belongs to exactly k circles. It is not known does such configurations exist for $k > 5$.

8. (A.Erdnigor) Let $ABCA'B'C'$ be a centrosymmetric octahedron (vertices A and A', B and B', C and C' are opposite) such that the sums of four planar angles equal $240°$ for each vertex. The Torricelli points T_1 and T_2 of triangles ABC and $A'BC$ are marked. Prove that the distances from T_1 and T_2 to BC are equal.

   Solution. Let D be the vertex of a parallelogram $AB'CD$. Then the faces of tetrahedron $ABCD$ are congruent to the faces of the octahedron, and the sums of four angles opposite to two non-intersecting edges (for example, $\angle CAD + \angle CBD + \angle ACB + \angle ADB$) equal $240°$. Let A_1, B_1, C_1, D_1 be the touching points of the insphere with the faces BCD, CDA, DAB, ABC

respectively. Then the triangles A_1BC and D_1BC are congruent, and this is also correct for five similar pairs of triangles. Therefore, $\angle BD_1C+\angle BA_1C = \angle BAC + \angle ABD_1 + \angle ACD_1 + \angle BDC + \angle DCA_1 + \angle DBA_1 = \angle BAC + \angle BDC + \angle ABC_1 + \angle ACB_1 + \angle DCB_1 + \angle DBC_1 = 240^\circ$ and $\angle BD_1C = \angle BA_1C = 120^\circ$. Similarly $\angle AD_1B = \angle AD_1C = \angle BA_1C = \angle BA_1D = 120^\circ$, i.e. A_1, D_1 coincide with the Torricelli points, which yields the required assumption.

Remark. Tetrahedrons with Torricelli points coinciding with the touching points of the insphere are called *isogonal* or *Gergonian*. It is known that the segments joining the Torricelli points with the opposite vertices of such tetrahedrons are concurrent, and the products of cosines of a halves of opposite bihedral angles are equal. The problem gives another characteristic property of Gergonian tetrahedrons.