X Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2014, July 31

Solutions

First day. 8 grade

8.1. (*J. Zajtseva, D. Shvetsov*) The incircle of a right-angled triangle ABC touches its catheti AC and BC at points B_1 and A_1, the hypotenuse touches the incircle at point C_1. Lines C_1A_1 and C_1B_1 meet CA and CB respectively at points B_0 and A_0. Prove that $AB_0 = BA_0$.

First solution. Consider an excircle with center I_A touching side AC at point B_2 and the extension of side BC at point A'_0. Since $I_AB_2CA'_0$ is a square, we have $I_AA'_0 = B_2C$. It is known that $B_2C = AB_1$, thus $I_AA'_0 = AB_1$. Then $A'_0B_1 \parallel I_AA$, but $I_AA \parallel B_1C_1$, therefore, A'_0, B_1, C_1 are collinear and A'_0 coincides with A_0, thus BA_0 as a tangent to the excircle is equal to the semiperimeter of ABC. Similarly we obtain that AB_0 is equal to the semiperimeter, therefore $AB_0 = BA_0$.

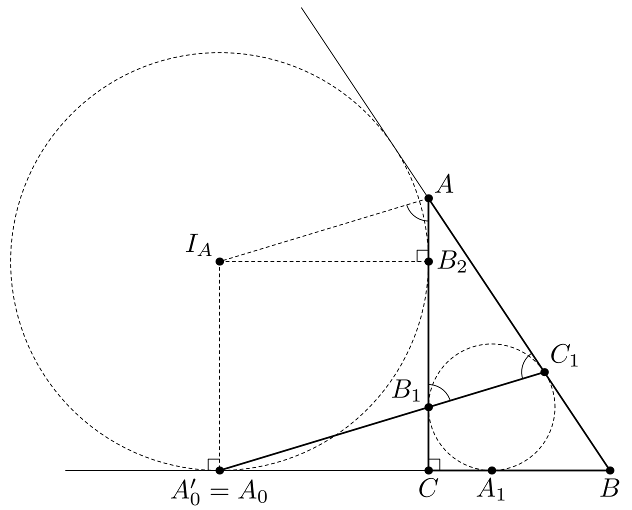

Fig. 8.1

Second solution. Since segments CA_1 and CB_1 are equal to the radius r of the incircle, and lines C_1A_1, C_1B_1 are perpendicular to the bisectors of angles B and A respectively, we obtain from right-angled triangles CA_0B_1 and CB_0A_1 that $A_0C = \frac{r}{\tan\frac{A}{2}}$, $B_0C = \frac{r}{\tan\frac{B}{2}}$. On the other hand $AC = r + \frac{r}{\tan\frac{A}{2}}$, $BC = r + \frac{r}{\tan\frac{B}{2}}$. Therefore $AB_0 = AC + CB_0 = BC + CA_0 = BA_0$.

8.2. (*B. Frenkin*) Let AH_a and BH_b be altitudes, AL_a and BL_b be angle bisectors of a triangle ABC. It is known that $H_aH_b \parallel L_aL_b$. Is it necessarily true that $AC = BC$?

Answer: yes.

First solution. Since triangles H_aH_bC and ABC are similar, triangles L_aL_bC and ABC are also similar, i.e $L_aC/AC = L_bC/BC$. Thus triangles AL_aC and BL_bC are similar. Thus, $\angle L_aBL_b = \angle L_bAL_a$, but these angles are equal to the halves of angles A and B. Therefore $AC = BC$.

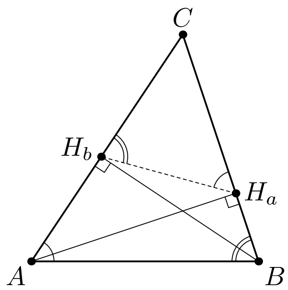

Fig. 8.2a

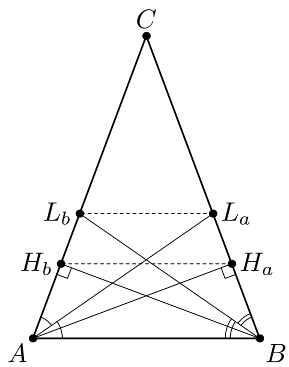

Fig. 8.2b

Second solution. Since H_aH_b and AB are antiparallel wrt AC and BC, L_aL_b and AB are also antiparallel wrt AC and BC, thus quadrilateral AL_bL_aB is cyclic. Then $\angle L_aBL_b = \angle L_bAL_a$ and $AC = BC$.

8.3. (*A. Blinkov*) Points M and N are the midpoints of sides AC and BC of a triangle ABC. It is known that $\angle MAN = 15^\circ$ and $\angle BAN = 45^\circ$. Find the value of angle ABM.

Answer: 75°.

First solution. Extend segment MN and consider such points K and L that $KM = MN = NL$ (fig. 8.3Ю). Since M is the midpoint of segments AC and KN, we obtain that $AKCN$ is a parallelogram. then $\angle CKM = 45^\circ$, $\angle KCM = 15^\circ$. Consider such point P on segment CM that $\angle CKP = 15^\circ$. Segment

KP divides triangle KCM into two isosceles triangles. Also $\angle PMN = 60°$, hence triangle MPN is regular. Triangles PLN and PKM are equal, triangle CPL is isosceles and right-angled, thus $\angle CLN = \angle CLP + \angle MLP = 75° = \angle ABM$, because $CLBM$ is a parallelogram.

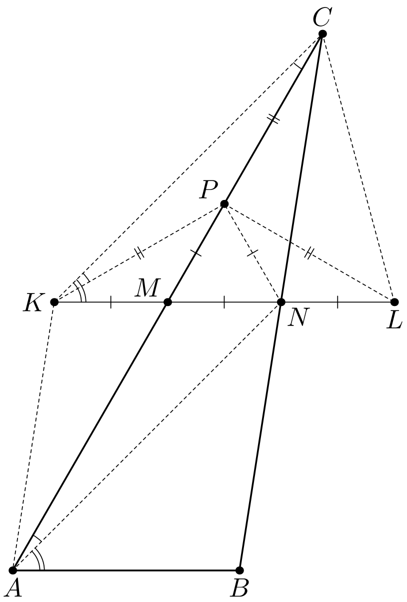

Fig. 8.3a

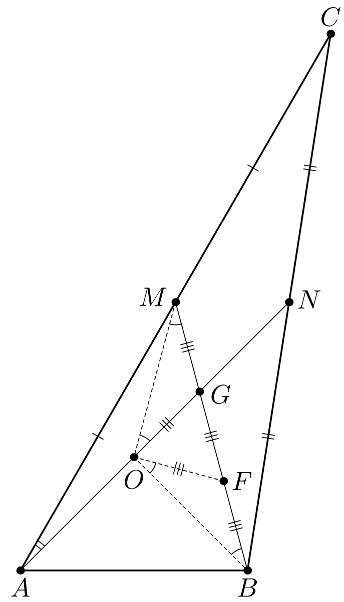

Fig. 8.3b

Second solution. Let G be the centroid of ABC, F be the midpoint GB, and GFO be the regular triangle such that points O and A lie in the same semiplane wrt MB. Since $\angle MOB = 120°$, O is the circumcenter of triangle MAB, also we have $\angle MOG = 30° = 2\angle MAG$, therefore AG meet OG on the circumcircle of AMB, i.e. A, O, G are collinear. Then $75° = \angle MOA/2 = \angle ABN$.

8.4. (*T. Kazitsyna*) Tanya has cut out a triangle from checkered paper as shown in the picture. The lines of the grid have faded. Can Tanya restore them without any instruments only folding the triangle (she remembers the triangle sidelengths)?

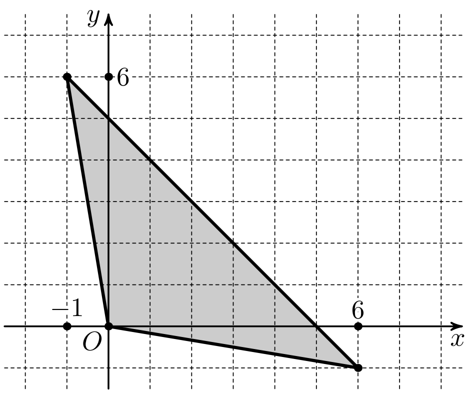

Fig. 8.4a

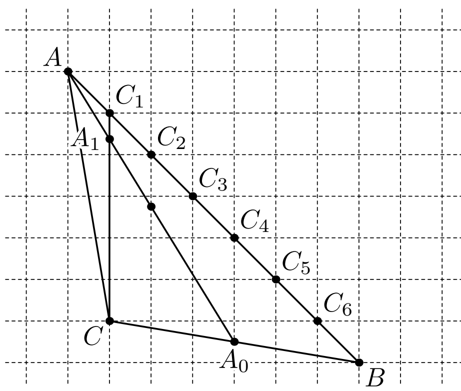

Fig. 8.4b

Solution. Let ABC be the given triangle ($AC = BC$). It is evident that we can find the midpoint of an arbitrary segment. Construct the median AA_0, and find on it such point A_1 that $AA_1 = AA_0/4$. By Thales theorem line CA_1 is the grid line intersecting AB at point C_1 such that $AC_1 = AB/7$ (fig.). Now constructing segments $C_1C_2 = C_2C_3 = \cdots = C_5C_6 = AC_1$, we find all nodes lying on AB. Folding the triangle by the line passing through C_2 in such way that C_3 be on CC_1, we restore the grid line passing through C_2, etc. The perpendicular lines can be restored similarly.

Solutions

Second day. 8 grade

8.5. (*A. Shapovalov*) A triangle with angles of 30, 70 and 80 degrees is given. Cut it by a straight line into two triangles in such a way that an angle bisector in one of these triangles and a median in the other one drawn from two endpoints of the cutting segment are parallel to each other. (It suffices to find one such cutting.)

Solution. Let in triangle ABC $\angle A = 30°$, $\angle B = 70°$, $\angle C = 80°$. Take an altitude AH. Then $\angle CAH = \angle MHA = 10°$, where M is the midpoint of AC. Also $\angle HAL = 10°$, where L is the foot of the bisector of triangle HAB from vertex A. Therefore the median of triangle AHC from H and the bisector of triangle BAH from A are parallel, and AH is the desired cutting segment.

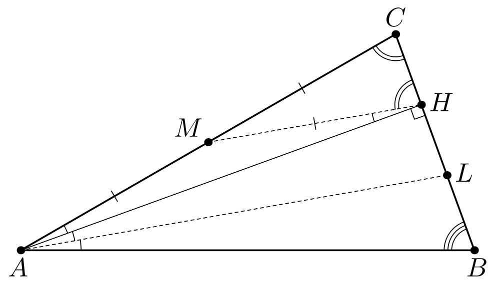

Fig. 8.5

8.6. (*V. Yasinsky*) Two circles k_1 and k_2 with centers O_1 and O_2 are tangent to each other externally at point O. Points X and Y on k_1 and k_2 respectively are such that rays O_1X and O_2Y are parallel and codirectional. Prove that two tangents from X to k_2 and two tangents from Y to k_1 touch the same circle passing through O.

Solution. Let S be the common point of XO_2 and YO_1. Let r_1 and r_2 be the radii of the corresponding circles. Then $\frac{XS}{SO_2} = \frac{O_1S}{SY} = \frac{r_1}{r_2} = \frac{O_1O}{OO_2}$. Thus $SO = \frac{r_1}{r_1+r_2}O_2Y = \frac{r_1r_2}{r_1+r_2}$.

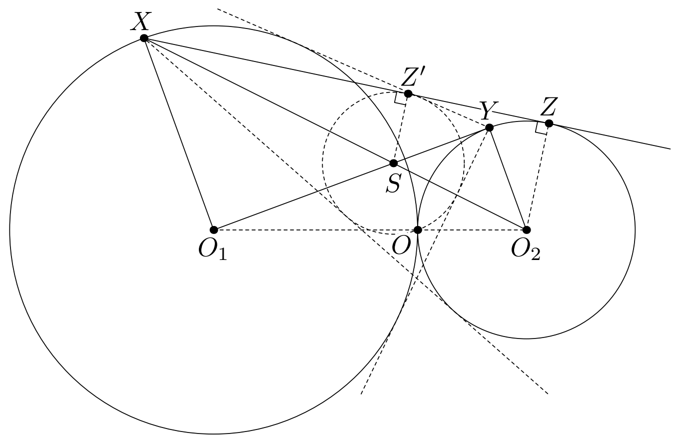

Fig. 8.6

Let XZ be a tangent from X to, and Z' be the projection of S to XZ. Then $SZ' = \frac{r_1}{r_1+r_2}O_2Z = \frac{r_1r_2}{r_1+r_2} = SO$. Similarly the distance from S to three remaining tangents is equal to SO, i.e. S is the center of the desired circle.

8.7. (*Folklor*) Two points on a circle are joined by a broken line shorter than the diameter of the circle. Prove that there exists a diameter which does not intersect this broken line.

Solution. Let A and B be the endpoints of the broken line. Consider the diameter XY parallel to AB. Let C be the reflection of B in XY, then AC is a diameter of the circle. Consider an arbitrary point Z on XY. Since $AZ + BZ = AZ + CZ \geq AC$, Z can not lie on the broken line, therefore XY is the desired diameter.

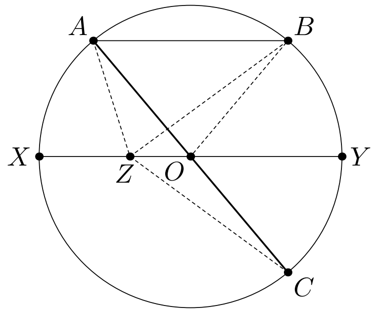

Fig. 8.7

8.8. (*Tran Quang Hung*) Let M be the midpoint of the chord AB of a circle centered at O. Point K is symmetric to M with respect to O, and point P is chosen arbitrarily on the circle. Let Q be the intersection of the line perpendicular to AB through A and the line perpendicular to PK through P. Let H be the projection of P onto AB. Prove that QB bisects PH.

First solution Let QA intersect the circle (O) at C which is distinct from A. Since BC is the diameter of the circle (O), we obtain that BC and MK bisect each other at the center of the circle, which implies that the quadrilateral $CKBM$ is a parallelogram. Furthermore, M is the midpoint of AB, then $CKMA$ is a rectangle since one of its angles is right. We shall prove that MQ is perpendicular to PC. We have

$$MC^2 - MP^2 - QC^2 + QP^2 = (CK^2 + MK^2) - (2PO^2 + 2OK^2 - PK^2) - (QK^2 - CK^2) + \\ + (QK^2 - PK^2) = 2CK^2 + 4OK^2 - 2PO^2 - 2OK^2 = 2CK^2 + 2OK^2 - 2OC^2 = 0.$$

Hence, MQ is perpendicular to PC. Let BP meet QA at R. Notice that CB is a diameter of (O), then BR is perpendicular to PC. Thus, it follows that MQ is parallel to BR. Q is the midpoint of AR, which follows from the fact that M is the midpoint of AB. Hence, QB bisects PH.

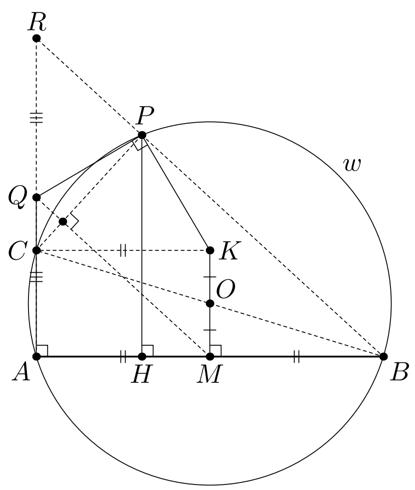

Fig. 8.8a

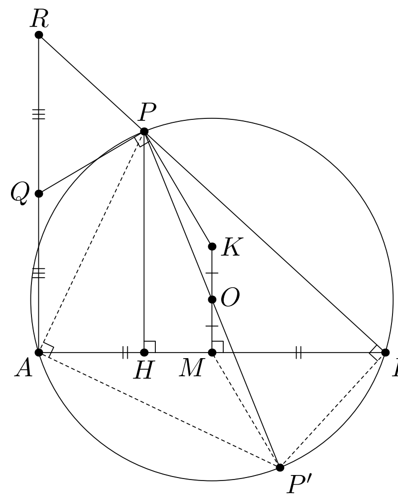

Fig. 8.8b

Second solution. Note that $\angle PBA \neq 90°$; in the other case $PK \parallel AB$, and point Q doesn't exist. Then BP meets AQ at point R. Since triangles BPH and BRA are homothetic, we have to prove that Q is the midpoint of AR.

Let point P' be opposite to P. Then $PA \perp P'A$, $PR \perp P'B$, $AR \perp AB$, i.e. the correspondent sides of triangles $P'AB$ and PAR are perpendicular. Thus these triangles are similar and their medians from P and P' are also perpendicular. Using the symmetry wrt O we obtain that $P'M \parallel PK \perp PQ$. Therefore PQ is the median in $\triangle PAR$.

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Final round. Ratmino, 2014, July 31

Solutions
First day. 9 grade

9.1. (*V. Yasinsky*) Let $ABCD$ be a cyclic quadrilateral. Prove that $AC > BD$ if and only if

$$(AD - BC)(AB - CD) > 0.$$

First solution. Without loss of generality we can suppose that arcs ABC and BCD are not greater than a cemicircle. Then $\smile AD = 2\pi - \smile ABC - \smile BCD + \smile BC > \smile BC$. Since arc $ABCD$ is also greater than arc BC, we obtain that $AD > BC$.

Now if $AC > BD$, then $\smile ABC > \smile BCD$, $\smile AB > \smile CD$ and $AB > CD$. If $AC < BD$ all inequalities are opposite.

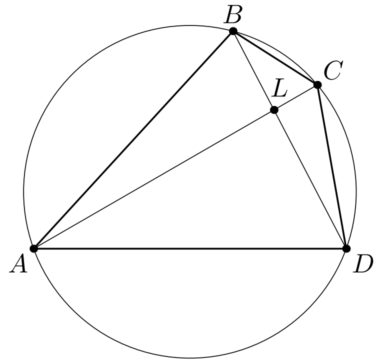

Fig. 9.1

Second solution. Let M, N be the midpoints of AC and BD, L be their common point, and O be the circumcenter. Let AL be the longest of segments AL, BL, CL, DL. Since $AL \cdot CL = BL \cdot DL$, CL is the shortest of these segments. Then $LM > LN$, $OM < ON$ and $AC > BD$. Also since triangles ALB and DLC are similar we obtain that $\dfrac{AB}{CD} = \dfrac{AL}{DL}$, i.e $AB > CD$. By the same way using the similarity of triangles ALD and BLC we obtain $AD > BC$.

Third solution. Note that $AC = 2R\sin B$ and $BD = 2R\sin A$, thus inequality $AC > BD$ is equivalent to $\sin B > \sin A$.

Now $(AD - BC)(AB - CD) > 0 \Leftrightarrow AD \cdot AB + BC \cdot CD > AD \cdot CD + BC \cdot AB$, which is equivalent to (multiply to $\frac{1}{2}\sin A \sin B = \frac{1}{2}\sin A \sin D = \frac{1}{2}\sin C \sin B$).

$$\left(\frac{AD \cdot AB \sin A}{2} + \frac{BC \cdot CD \sin C}{2}\right)\sin B > \left(\frac{AD \cdot CD \sin D}{2} + \frac{BC \cdot AB \sin B}{2}\right)\sin A \Leftrightarrow$$

$$\Leftrightarrow (S(DAB) + S(BCD))\sin B > (S(CDA) + S(ABC))\sin A \Leftrightarrow$$

$$\Leftrightarrow S(ABCD)\sin B > S(ABCD)\sin A \Leftrightarrow \sin B > \sin A.$$

9.2. (*F. Nilov*) In a quadrilateral $ABCD$ angles A and C are right. Two circles with diameters AB and CD meet at points X and Y. Prove that line XY passes through the midpoint of AC.

Solution. Let M, N, K be the midpoints of AB, CD and AC respectively. Then the degree of point K wrt the circle with diameter AB is equal to $KM^2 - MA^2 = \dfrac{CB^2 - AB^2}{4}$, and its degree wrt the circle with diameter CD is equal to $\dfrac{AD^2 - CD^2}{4}$. Since $AB^2 + AD^2 = BD^2 = BC^2 + CD^2$, we obtain that these degrees are equal.

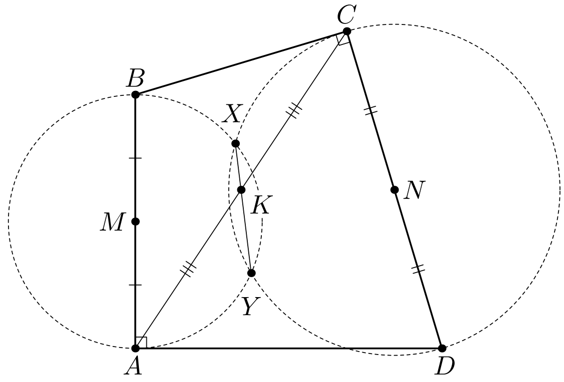

Fig. 9.2

9.3. (*E. Diomidov*) An acute angle A and a point E inside it are given. Construct points B, C on the sides of the angle such that E is the center of the Euler circle of triangle ABC.

First solution. Let l_1 and l_2 be the arms of $\angle A$ so that rotating l_1 about A to an angle $\alpha < 90^\circ$ maps it onto l_2. Rotate l_2 about E to an angle 2α and let its image meet l_1 at M_b and B be the reflection of A in M_a. The vertex C is constructed analogously.

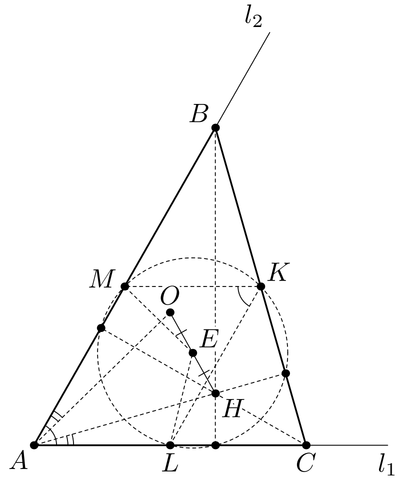

Fig. 9.3

Second solution. Let O and H be the circumcenter and the orthocenter of the sought triangle. Then E is the midpoint of OH, $\angle BAO = \angle HAC$ and $AH = 2AO\cos\angle A$. Therefore the composition of the reflection about the bisector of angle A, the homothety with center A and the coefficient equal to $2\cos\angle A$ and the reflection around E is a similarity with center O. Thus finding the center of this similarity we can construct B and C as the second common points of the arms of the given angle and the circle with center O, passing through A.

Note. If $\angle A = 60^\circ$ the considered similariry is the reflection about the line passing through E and perpendicular to the bisector of angle A. Thus we can take as O an arbitrary point of this line. In the other cases the solution is unique.

9.4. (*Mahdi Etesami Fard*) Let H be the orthocenter of a triangle ABC. Given that H lies on the incircle of ABC , prove that three circles with centers A, B, C and radii AH, BH, CH have a common tangent.

First solution. Let H_a, H_b, H_c be the feet of the altitudes. Since $AH \cdot HH_a = BH \cdot HH_b = CH \cdot HH_c$, there exists an inversion about a circle with center H, transforming A, B, C to H_a, H_b, H_c respectively (if the triangle is acute-angled take a composition of the inversion and the reflection around H). This inversion transforms the sidelines of the triangle to the circles with diameters AH, BH, CH, and it transforms the incircle to the line touching these three circle. The homothety with center H and the coefficient 2 transforms this line to the sought one.

Second solution. Let I be the center of the incircle, A_1, B_1, C_1 be its touching points with BC, AC, AB respectively, and A_2, B_2, C_2 be such points on three circles that $\triangle A_1IH \sim \triangle HAA_2$, $\triangle B_1IH \sim \triangle HBB_2$

and $\triangle C_1IH \sim \triangle HCC_2$. The tangents to the circles in these points and the tangent to the incircle in H are parallel; prove that these three tangents coincide, i.e. the projections of vectors $\overrightarrow{HA_2}$, $\overrightarrow{HB_2}$ and $\overrightarrow{HC_2}$ to IH are equal. It is evident that they are codirectional. Since the angles formed by HA_2 with IH and IA_1 are equal, the first projection are equal to the projection of HA_2 to AH, i.e. $\frac{AH}{r} \cdot HH_a$. Find similarly the remaining projections and note that $AH \cdot HH_a = BH \cdot HH_b = CH \cdot HH_c$.

Solutions
Second day. 9 grade

9.5. (*D. Shvetsov*) In triangle ABC $\angle B = 60^\circ$, O is the circumcenter, and L is the foot of an angle bisector of angle B. The circumcirle of triangle BOL meets the circumcircle of ABC at point $D \neq B$. Prove that $BD \perp AC$.

Solution. Let H be the orthocenter of ABC, and D' be the reflection of H in AC. Then D' lies on the circumcircle, and since $\angle B = 60^\circ$, we have $BO = BH$. Thus, since BL is the bisector of angle OBH, then $LO = LH = LD'$. Therefore $BOLD'$ is a cyclic quadrilateral, i.e. D' coincides with D.

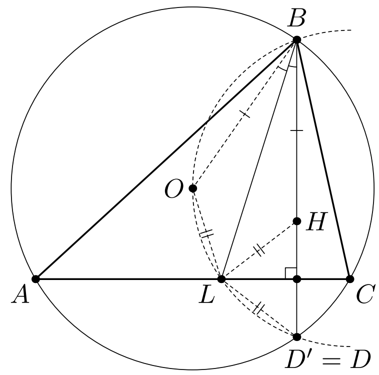

Fig. 9.5

9.6. (*A. Polyansky*) Let I be the incenter of triangle ABC, and M, N be the midpoints of arcs ABC and BAC of its circumcircle. Prove that points M, I, N are collinear if and only if $AC + BC = 3AB$.

First solution. Let A_1, B_1, C_1 be the midpoints of arcs BC, CA, AB of the circumcircle, not containing the other vertices of ABC. It is evident that MN and A_1B_1 are equal and parallel. Therefore they cut equal segments CC_2 and IC_1, where C_2 is the midpoint of CI, on the line CC_1, perpendicular to MN. Since C_1 is the circumcenter of triangle AIB we obtain that $C_2A_0 = C_2C = IC_1 = C_1A = C_1B$ (A_0 and B_0 are the touching points of the incircle with BC and CA respectively). Thus triangles C_2CA_0 and C_1AB are equal ($AB = CA_0$). From this $AC + CB = AB_0 + B_0C + CA_0 + A_0B = 2AB + AB_0 + A_0B = 3AB$. Similarly we obtain the opposite assertion.

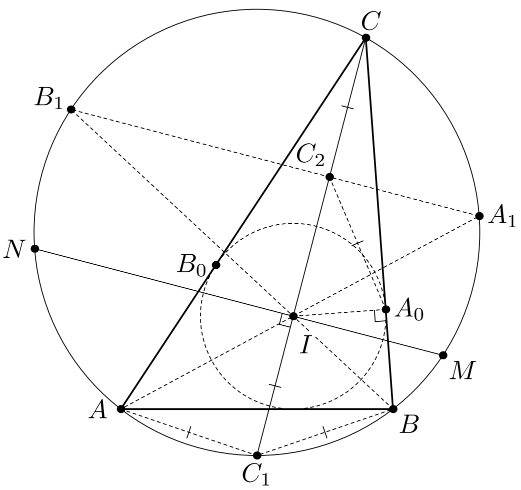

Fig. 9.6a

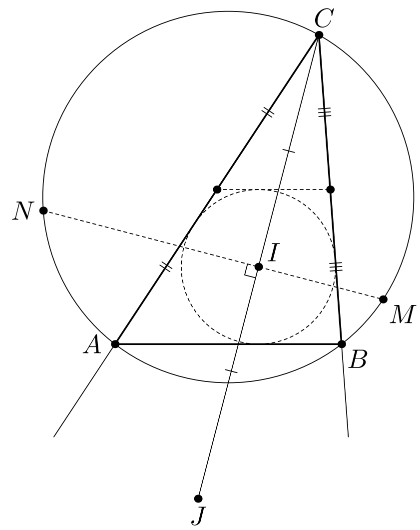

Fig. 9.6b

Second solution. Let J be the center of the excircle touching side AB. Then M and N are the centers of circles ACJ and BCJ, and therefore MN is the perpendicular bisector to segment CJ, i.e. I is the midpoint of CJ. Using the homothety with center C and the coefficient $1/2$ we obtain that the incircle touches the medial line parallel to AB. The trapezoid formed by this medial line and the sidelines of ABC is circumscribed if the sought equality is correct.

9.7. (*N. Beluhov*) Nine circles are drawn around an arbitrary triangle as in the figure. All circles tangent to the same side of the triangle have equal radii. Three lines are drawn, each one connecting one of the triangle's vertices to the center of one of the circles touching the opposite side, as in the figure. Show that the three lines are concurrent.

Solution. Introduce the following notation. Let r_a, r_b, r_c be the radii of the circles centered at O_a, O_b, O_c, respectively. Let $d_a(X)$ be the distance from X to BC, and define d_b and d_c analogously.

The figure composed of the lines CA and CB and the first three circles in the chain tangent to CA, counting from C, is similar to the figure composed of the lines CB and CA and the chain tangent to CB. Therefore, $d_a(O_b) : r_b = d_b(O_a) : r_a$. Analogous reasoning applies to the vertices A and B.

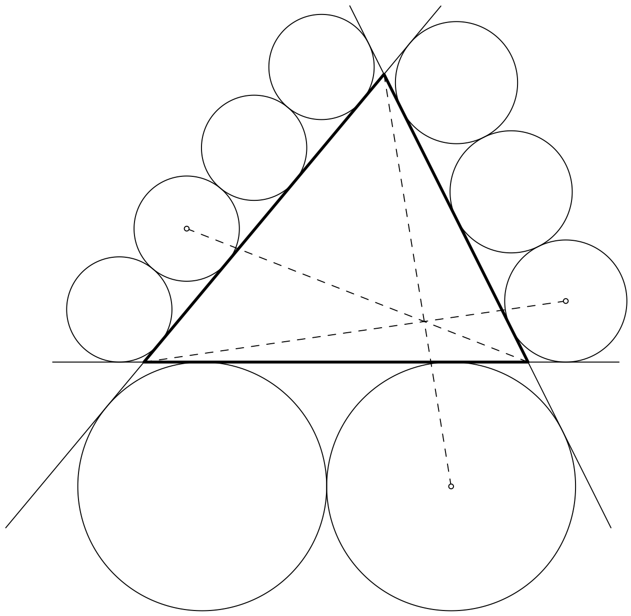

Fig. 9.7a

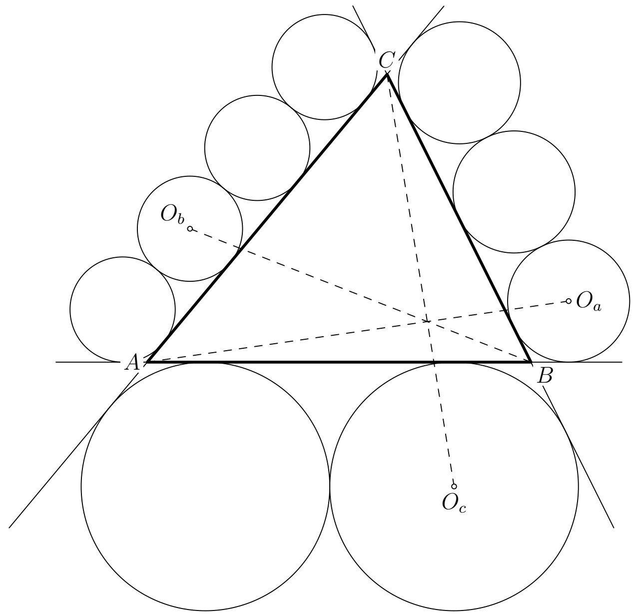

Fig. 9.7b

We have, therefore,

$$\frac{d_c(O_a)}{d_b(O_a)} \cdot \frac{d_a(O_b)}{d_c(O_b)} \cdot \frac{d_b(O_c)}{d_a(O_c)} = \frac{r_a}{r_c} \cdot \frac{r_c}{r_b} \cdot \frac{r_b}{r_a} = 1,$$

and the claim follows.

9.8. (*N. Beluhov, S. Gerdgikov*) A convex polygon P lies on a flat wooden table. You are allowed to drive some nails into the table. The nails must not go through P, but they may touch its boundary. We say that a set of nails blocks P if the nails make it impossible to move P without lifting it off the table. What is the minimum number of nails that suffices to block any convex polygon P?

Solution. If P is a parallelogram, then you need at least four nails to block it. Indeed, if there is a side s of P such that no nail touches the interior of s, then you can slide P in the direction determined by the two sides adjacent to s.

Now let P be an arbitrary convex polygon. We will show that four nails suffice to block P.

A set of nails blocks P if and only if, for every sufficiently small movement f (i.e., for every translation to a sufficiently small distance and every rotation to a sufficiently small angle), the interior of the image $f(P)$ of P covers some nail.

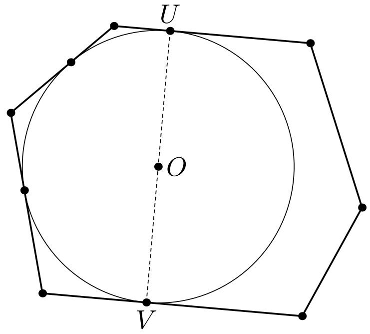

Fig. 9.8a

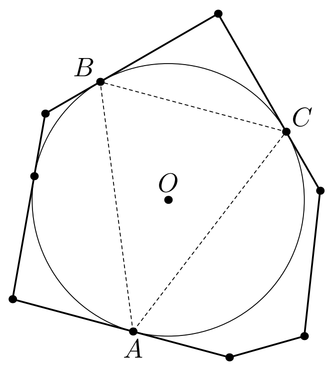

Fig. 9.8b

Let the circle c of center O be one of the largest circles contained within P. Let A_1, A_2, ... A_k be the points at which c touches P's boundary, and let H be their convex hull.

Suppose that there are two vertices U and V of H such that UV is a diameter of c. Place two nails at U and V. It is easy to see that, since the sides of P that contain U and V are parallel, the only movements still permitted to P are the translations in a direction perpendicular to UV. (Indeed, all other directions of translation would cause P to cover either U or V when the translation distance is small enough; all clockwise rotations whose center lies to the left of the ray $\overrightarrow{UV}$ would cause P to cover V when the rotation angle is small enough; all clockwise rotations whose center lies to the right of $\overrightarrow{UV}$ would cause P to cover U when the rotation angle is small enough; and so on.) A third nail prevents P from sliding to the left of $\overrightarrow{UV}$, and a fourth one prevents it from sliding to the right.

We are left to consider the case when no side or diagonal of H contains O.

Suppose that $O \notin H$. Let PQ be that side of H which separates H and O and let the tangents to c at P and Q meet in T. Then a homothety of center T and ratio larger than and sufficiently close to one maps c onto a larger circle contained within P: a contradiction.

Therefore, $O \in H$. Consider an arbitrary triangulation π of H and let ABC be that triangle in π which contains O. (A, B, and C being three of the contact points of H with the boundary of P.)

Since no side or diagonal of H contains O, O lies in the interior of $\triangle ABC$. It is easy to see, then — as above — that three nails placed at A, B, and C block P.

X Geometrical Olympiad in honour of I.F.Sharygin
Final round. Ratmino, 2014, July 31

Solutions
First day. 10 grade

10.1. (*I. Bogdanov, B. Frenkin*) The vertices and the circumcenter of an isosceles triangle lie on four different sides of a square. Find the angles of this triangle.

Answer. 15°, 15° and 150°.

Solution. Let the circumcenter O of triangle XYZ lie on side AB, and its vertices X, Y, Z lie on sides BC, CD, DA of square $ABCD$. Since segment OY intersect segment XZ, angle XYZ is obtuse, thus XZ is the base of the triangle. Then $OY \perp XZ$; since segments OY and XZ are perpendicular and their projections to perpendicular lines BC and AB respectively are equal, we obtain that these segments are also equal, i.e. the side of the triangle is equal to its circumradius. Since angle XYZ is obtuse, we obtain that $\angle XYZ = 150^\circ$, then two remaining angles are equal to 15°.

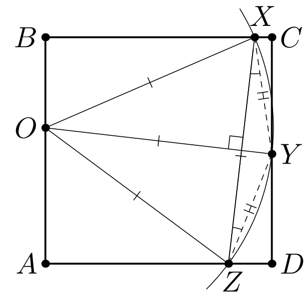

Fig. 10.1

10.2. (*A. Zertsalov, D. Skrobot*) A circle, its chord AB and the midpoint W of the minor arc AB are given. Take an arbitrary point C on the major arc AB. The tangent to the circle at C meets the tangents at A and B at points X and Y respectively. Lines WX and WY meet AB at points N and M respectively. Prove that the length of segment NM does not depend on point C.

First solution. Let T be the common point of AB and CW. Then AT and AC are antiparallel wrt angle AWC. Since WX is the symedian of triangle CAW, it is the median of triangle ATW, Thus N is the midpoint of AT. Similarly M is the midpoint of BT, i.e. $MN = AB/2$.

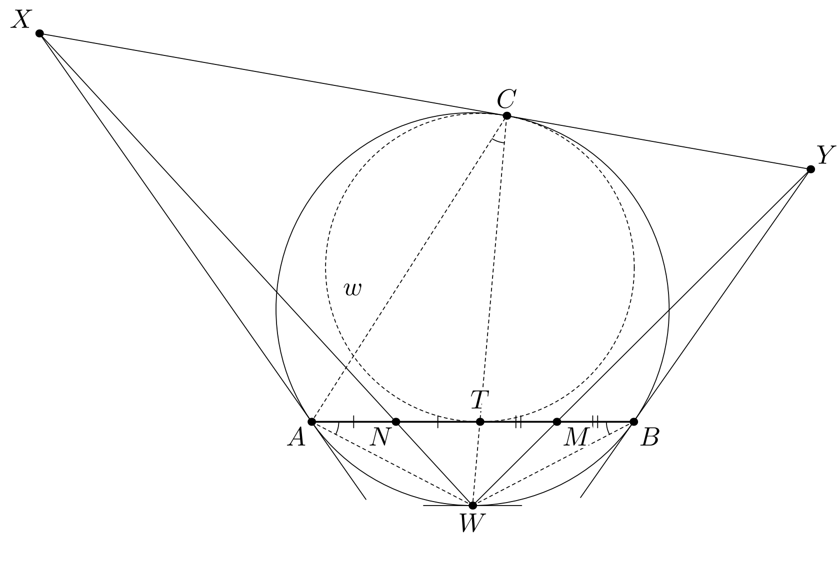

Рис. 10.2

Second solution. Consider circle w, touching XY at C and touching AB (at point T). It is easy to see that WX is the radical axis of A and w, i.e. it passes through the midpoint N of segment AT, Similarly WY passes through the midpoint M of segment ZB. Thus $MN = AB/2$.

10.3. (*A. Blinkov*) Do there exist convex polyhedra with an arbitrary number of diagonals (a *diagonal* is a segment joining two vertices of a polyhedron and not lying on the surface of this polyhedron)?

Answer. Yes.

Solution. Let $SA_1 \dots A_{n+2}$ be a $(n+2)$-gon pyramid and $TSA_{n+1}An+2$ be a pyramid with base $SA_{n+1}An+2$ and sufficiently small altitude. Then the diagonals of polyhedron $TSA_1 \dots A_{n+2}$ are segments $TA_1, \dots, TA_n$.

10.4. (*A. Garkavyj, A. Sokolov*) Let ABC be a fixed triangle in the plane. Let D be an arbitrary point in the plane. The circle with center D, passing through A, meets AB and AC again at points A_b and A_c respectively. Points B_a, B_c, C_a and C_b are defined similarly. A point D is called *good* if the points A_b, A_c, B_a, B_c, C_a, and C_b are concyclic. For a given triangle ABC, how many good points can there be?

Answer. 4.

Solution. It is evident that the circumcenter O satisfies the condition. Now let D does not coincide with O. Let A', B', C' be the projections of D to BC, CA, AB respectively. Then the midpoints of segments AB and A_bB_a are symmetric wrt C', therefore the perpendicular bisector to A_bB_a passes through point O', symmetric to O wrt D. The perpendicular bisectors to A_cC_a and B_cC_b also pass through O', thus O' is the center of the circle passing through six points.

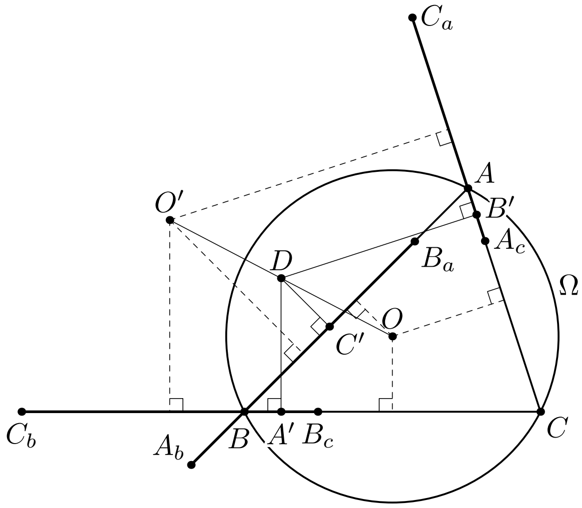

Fig. 10.4a

Since points D and O' are on equal distances from A_b and A_c, line DO' is the perpendicular bisector to A_bA_c. But $A_bA_c \parallel B'C'$, therefore $DO' \perp B'C'$. Similarly $DO' \perp A'B'$, i.e. points A', B', C' are collinear. Thus, D lies on the circumcircle of ABC and its Simson line $A'B'C'$ is perpendicular to radius OD. When D moves on the circle its Simson line rotates in the opposite direction with twice as smaller velocity, therefore there exists exactly three points with such property (these points form a regular triangle).

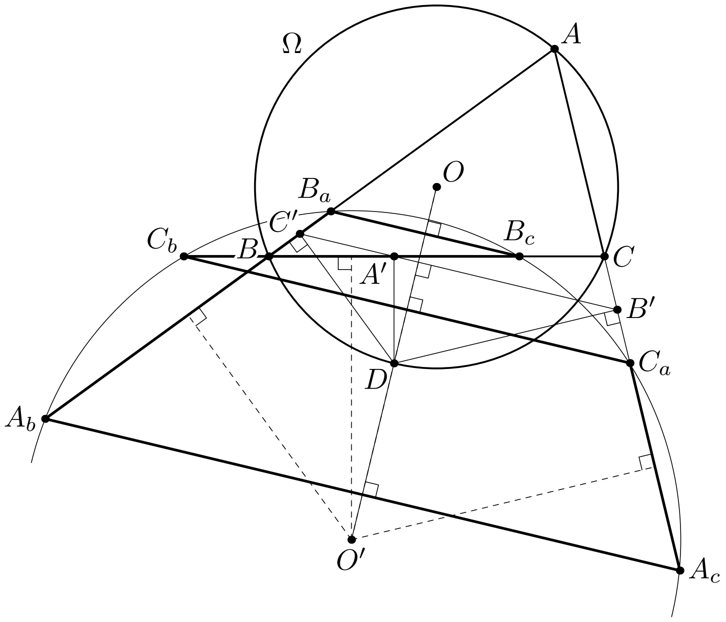

Fig. 10.4b

But several of these points can coincide with the vertices of the given triangle. Since the Simson line of the vertex A coincide with the corresponding altitude, that happens when the radius OA is parallel to BC, i.e. $|\angle B - \angle C| = 90°$. This is true for two vertices iff the angles of the given triangle are equal to $30°$, $30°$ and $120°$. From this the answer follows.

X Geometrical Olympiad in honour of I.F.Sharygin
Final round. Ratmino, 2014, August 1

Solutions
Second day. 10 grade

10.5. (*A. Zaslavsky*) The altitude from one vertex of a triangle, the bisector from the another one and the median from the remaining vertex were drawn, the common points of these three lines were marked, and after this everything was erased except three marked points. Restore the triangle. (For every two erased segments, it is known which of the three points was their intersection point.)

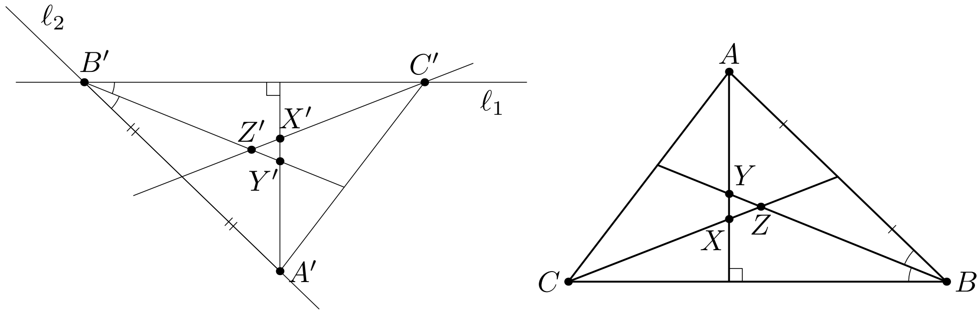

Fig. 10.5

Solution. Let X, Y, Z be the marked points. Then we have to find points A, B, C on lines XY, YZ, ZX respectively such that XY, YZ, ZX be the altitude, the bisector and the median of triangle ABC. From an arbitrary point B' draw a ray l_1 perpendicular to XY, and such ray l_2, that the bisector of the angle formed by these rays be parallel to YZ. Take an arbitrary point A' on l_2 and draw through the midpoint of $A'B'$ the line parallel to ZX meeting l_1 at point C'. Triangle $A'B'C'$ is homothetic to the desired one. Constructing the points corresponding to X, Y, Z, find the center and the coefficient of the homothety.

10.6. (*E. H. Garsia*) The incircle of a non-isosceles triangle ABC touches AB at point C'. The circle with diameter BC' meets the incircle and the bisector of angle B again at points A_1 and A_2 respectively. The circle with diameter AC' meets the incircle and the bisector of angle A again at points B_1 and B_2 respectively. Prove that lines AB, A_1B_1, A_2B_2 concur.

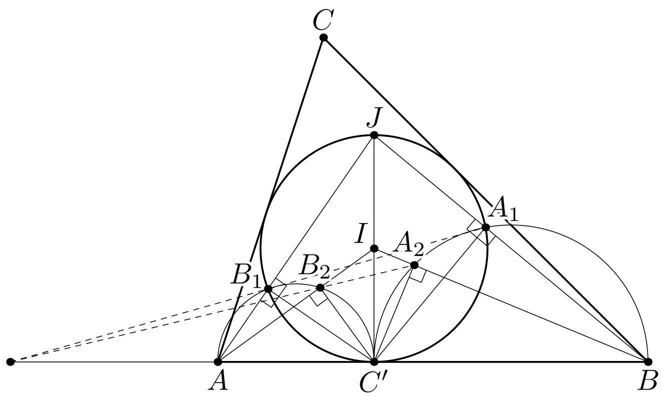

Fig. 10.6

Solution. Let I be the center of the incircle, and J be its point opposite to C'. Then A_1 and B_1 are the common points of AJ, BJ with the incircle (because $\angle AB_1C' = \angle C'B_1J = \angle BA_1C' = \angle C'A_1J = 90^\circ$). From right-angled triangles $AC'I$, $BC'I$, $AC'J$ and $BC'J$ with altitudes $C'B_2$, $C'A_2$, $C'B_1$ and $C'A_1$ we obtain

$$\frac{AB_2}{B_2I}\cdot\frac{IA_2}{A_2B} = \frac{AC'^2}{C'I^2}\cdot\frac{IC'^2}{C'B^2} = \frac{AC'^2}{C'J^2}\cdot\frac{JC'^2}{C'B^2} = \frac{AB_1}{B_1J}\cdot\frac{JA_1}{A_1B},$$

i.e. by Menelaos theorem A_1B_1 and A_2B_2 meet AB at the same point.

10.7. (*S. Shosman, O. Ogievetsky*) Prove that the smallest dihedral angle between faces of an arbitrary tetrahedron is not greater than the dihedral angle between faces of a regular tetrahedron.

Solution. Let the greatest area of the faces of the tetrahedron is equal to 1. Let S_1, S_2, S_3 be the areas of the remaining faces, and α_1, α_2, α_3 be the angles between these faces and the greatest face. Then $S_1 \cos\alpha_1 + S_2 \cos\alpha_2 + S_3 \cos\alpha_3 = 1$ and, therefore, one of angles α_1, α_2, α_3 is not greater than $\arccos\frac{1}{3}$.

10.8. (*N. Beluhov*) Given is a cyclic quadrilateral $ABCD$. The point L_a lies in the interior of $\triangle BCD$ and is such that its distances to the sides of this triangle are proportional to the lengths of corresponding sides. The points L_b, L_c, and L_d are defined analogously. Given that the quadrilateral $L_aL_bL_cL_d$ is cyclic, prove that the quadrilateral $ABCD$ has two parallel sides.

Solution. If $ABCD$ is an isosceles trapezoid, then so is $L_aL_bL_cL_d$.

Suppose, then, that $L_aL_bL_cL_d$ is cyclic and that $ABCD$ has no parallel sides. Let $P = AB \cap CD$, $Q = AD \cap BC$, and $R = AC \cap BD$. Furthermore, let the tangents at A and B to the circumcircle of $ABCD$ meet in S, those at B and C meet in T, those at C and D – in U, and those at D and A – in V.

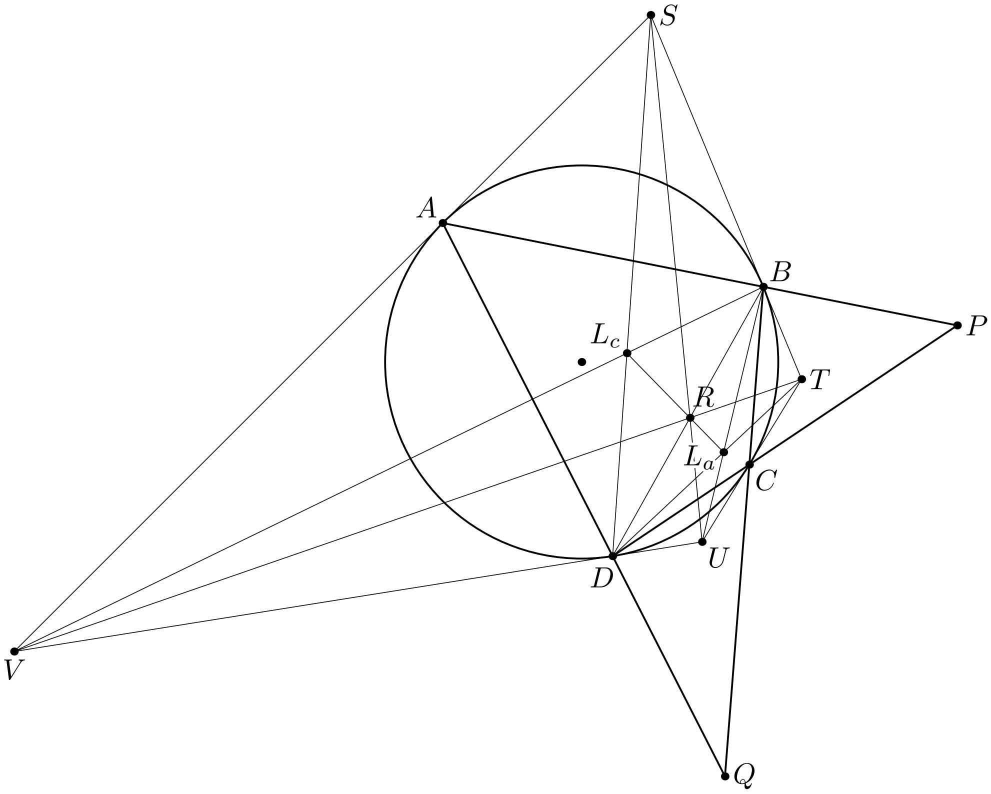

Fig. 10.8

It is well-known that $R = SU \cap TV$ and that $L_a = BU \cap DT$ and $L_c = BV \cap DS$. By Pappus's theorem for the hexagon $BUSDTV$, we see that R lies on L_aL_c. Similarly, R lies on L_bL_d and, therefore, $R = L_aL_c \cap L_bL_d$. Analogously, $P = L_aL_b \cap L_cL_d$ and $Q = L_aL_d \cap L_bL_c$.

Since the vertices of $\triangle PQR$ are the intersections of the diagonals and opposite sides of $ABCD$, the circumcircle k of $ABCD$ has the property that the polar of any vertex of $\triangle PQR$ with respect to k is the side opposite to that vertex. Analogously, the circumcircle s of $L_aL_bL_cL_d$ has the same property. Given $\triangle PQR$, however, there is exactly one such circle. It follows that $k \equiv s$, and this is a contradiction because $L_aL_bL_cL_d$ lies in the interior of $ABCD$.