Group Theory Problem List

1. (**CMI** 2017 PartA Problem-4) For a positive integer n, let Sn denote the permutation group on n symbols. Choose the
correct statement(s) from below:
(A) For every positive integer n and for every m with $1 \leq m \leq n, Sn$ has a cyclic subgroup of order m;
(B) For every positive integer n and for every m with $n < m < n!, Sn$ has a cyclic subgroup of order m;
(C) There exist positive integers n and m with $n < m < n!$ such that Sn has a cyclic subgroup of order m;
(D) For every positive integer n and for every group G of order n, G is isomorphic to a subgroup of Sn.
2. (**CMI** 2017 PartB Problem-15) For a group G, let $Aut(G)$ denote the group of group automorphisms of G. (The group operation of $Aut(G)$ is composition.) Let p be prime number. Show that the multiplicative group $\mathbb{F}_p/\{0\}$ is isomorphic to $Aut((\mathbb{F}_p, +))$ under the map $a \mapsto [b \mapsto ab]$ $(a\epsilon\mathbb{F}_p/\{0\}, b\epsilon\mathbb{F}_p)$.
3. (**CMI** 2016 Part B Problem-17) Let G be a non-trivial subgroup of the group $(\mathbb{R}, +)$. Show that either G is dense in $\mathbb{R}$ or that $G = \mathbb{Z} \cdot l \quad where \quad l = inf\{x\epsilon G | x > 0\}$.
4. (**CMI** 2014 Part A Problem-3) Let G be a finite group. An element $a\epsilon G$ is called a square if there exists $x\epsilon G$ such that $x^2 = a$. Which of the following statement(s) is/are true?
(A) If $a, b\epsilon G$ are not squares, ab is a square.
(B) Suppose that G is cyclic. Then if $a, b\epsilon G$ are not squares, ab is a square.

Consider the map $\phi : G \to G$ given by $\phi(a) = a^2$. Show that ϕ is not surjective.

6. (**CMI** 2013 PartA Problem-1) Pick the correct statement(s) below.
   (a) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
   (b) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/4$.
   (c) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and a subgroup isomorphic to $\mathbb{Z}/4$.
   (d) There exists a group of order 44 without any subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or to $\mathbb{Z}/4$.
7. (**CMI** 2013 PartA Problem-2) Let G be group. The following statements hold.
   (a) If G has nontrivial centre C, then G/C has trivial centre.
   (b) If $G \neq 1$, there exists a nontrivial homomorphism $h : \mathbb{Z} \to G$.
   (c) If $|G| = p^3$, for p a prime, then G is abelian.
   (d) If G is nonabelian, then it has a nontrivial automorphism.
8. (**CMI** 2013 PartB Problem-1) Let G be a finite group, p the smallest prime divisor of $|G|$, and $x \epsilon G$an element of order p. Suppose $h \epsilon G$ is such that $hxh^{-1} = x^{10}$. Show that $p = 3$.
9. (**CMI** 2012 PartA Problem-11) There are no infinite group with subgroups of index 5.
10. (**CMI** 2012 PartA Problem-12) Every finite group of odd order is isomorphic to a subgroup of A_n, the group of all even permutations.
11. (**CMI** 2011 PartA Problem-3 doubt) There is a continuous bijection from $\mathbb{R}^2 \to \mathbb{R}$.
12. (**CMI** 2011 PartA Problem-4 doubt) There is a bijection between $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$.
13. (**CMI** 2011 PartB- Problem3 doubt) Let S denote the group of all those permutations of the English alphabet that fix the letters T, E, N, D, U, L, K, A and R. Other letters may or may not be fixed. Show that S has elements σ, τ of order 36 and 39 respectively, but does not have any element of order 37 or 38.
14. (**CMI** 2011 PartB Problem-4 doubt) Show that there are at least two non-isomorphic groups of order 198. Show that in all those groups the number of elements of order 11 is the same.
15. (**ISI** 2017 **PMB** GroupB Problem-10) Determine all finite groups which have exactly 3 conjugacy classes.
16. (**ISI** 2016 **PMB** GroupB Problem-9) Let S_{17} be group of all permutations of 17 distinct symbols. How many subgroups of order 17 does S_{17} have? Justify your answer.
17. (**ISI** 2016 **PMB** GroupB Problem-10) Suppose that H and K are two subgroups of a group G. Assume that $[G : H] = 2$ and K is not a subgroup of H. Show that $HK = G$.
18. (**ISI** 2015 **PMB** GroupB Problem-4) Let G be a group which has only finitely many subgroups. Prove that G must be finite.
19. (**ISI** 2014 **PMB** GroupB Problem-1) Let $(\mathbb{Q}, +)$ be the group of rational numbers under addition. If G_1, G_2 are nonzero subgroups of $(\mathbb{Q}, +)$, then prove that $G_1 \cap G_2 \neq \{0\}$.
20. (**ISI** 2014 **PMB** GroupB Problem-2) With proper justifications, examine whether there exists any surjective
   group homomorphism
   (a) from the group $(\mathbb{Q}(\sqrt{2}), +)$ to the group $(\mathbb{Q}, +)$,
   (b) from the group $(\mathbb{R}, +)$ to the group $(\mathbb{Z}, +)$.

23. (**TIFR** 2018 Part A Problem-17) The multiplicative group $F_7^\times$ is isomorphic to a subgroup of the multiplicative group $F_{31}^\times$.
24. (**TIFR** 2018 Part A Problem-21) A countable group can have only countably many distinct subgroups.
25. (**TIFR** 2018 Part A Problem-23) The permutation group S_{10} has an element of order 30.
26. (**TIFR** 2018 Part B Problem-11) Consider a cube C centered at the origin in $\mathbb{R}^3$. The number of invertible linear transformations of $\mathbb{R}^3$ which map C onto itself is
    (a) 72.
    (b) 48.
    (c) 24.
    (d) 12.
27. (**TIFR** 2017Part I Problem-12) There exists a finite abelian group G containing exactly 60 elements of order 2.
28. (**TIFR** 2017Part I Problem-23) A p-Sylow subgroup of the underlying additive group of a finite commutative ring R is an ideal in R.
29. (**TIFR** 2017Part I Problem-27) In the symmetric group S_n any two elements of the same order are conjugate.
30. (**TIFR** 2017Part II Problem-3) Prove or disprove: the group of positive rationals under multiplication is isomorphic to its subgroup consisting of rationals which can be expressed as p/q, where both p and q are odd positive integers.
31. (**TIFR** 2017Part II Problem-7) Prove or disprove: If G is a finite group and $g, h \epsilon G$, then g, h have the same order if and only if there exists a group H containing G such that g and h are conjugate in H.
32. (**TIFR** 2016 Part-I Problem-8) The number of group homomorphisms from $\mathbb{Z}/20\mathbb{Z}$ to $\mathbb{Z}/29\mathbb{Z}$ is
    A.1
    B.20
    C.29
    D.580
33. (**TIFR** 2016 Part-I Problem-20) Let G = $\mathbb{Z}/100\mathbb{Z}$ and let S =$\{h \epsilon G : Order(h) = 50\}$. Then $|S|$ equals
    A. 20
    B. 25
    C. 30
    D. 50
34. (**TIFR** 2016 Part-II Problem-27) For $n \geq 1$, let S_n denote the group of all permutations on n symbols.
    Which of the following statements is true?
    A. S_3 has an element of order 4
    B. S_4 has an element of order 5
    C. S_4 has an element of order 6
    D. S_5 has an element of order 6.

A. $Aut(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$
B. If G is cyclic, then $Aut(G)$ is cyclic
C. If $Aut(G)$ is trivial, then G is trivial
D. $Aut(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

36. (**TIFR** 2015 Part II Problem-17) In how many ways can the group $\mathbb{Z}_5$ act on the set $\{1, 2, 3, 4, 5\}$?
A. 5
B. 24
C. 25
D. 120.
37. (**TIFR** 2015 Part II Problem-29) let G be a group. Suppose $|G| = p^2q$, where p and q are distinct prime numbers satisfying $q \not\equiv 1 \ mod\ p$. Which of the following is always true?
A. G has more than one p-Sylow subgroup
B. G has a normal p-Sylow subgroup
C.The number of q-Sylow subgroups of G is divisible by p
D. G has a unique q-Sylow subgroup.
38. (**NBHM** (PhD) 2017 Section 1 Problem-1.2) Let $n\epsilon\mathbb{N}, n \geq 2$. Which of the following statements are true?
a. Any finite group G of order n is isomorphic to a subgroup of $GL_n(\mathbb{R})$.
b. The group $\mathbb{Z}_n$ is isomorphic to a subgroup of $GL_2(\mathbb{R})$.
c. The group $\mathbb{Z}_{12}$ is isomorphic to a subgroup of S_7.
39. (**NBHM** (PhD) 2016 Section 1 Problem-1.3) Which of the following statements are true?
a. Let G be a group of order 99 and let H be a subgroup of order 11. Then H is normal in G.
b. Let H be the subgroup of S_3 consisting of the two elements $\{e, a\}$ where e is the identity and $a = (12)$. Then H is normal in S_3.
c. Let G be a finite group and let H be a subgroup of G. Define $W = \cap_{g\epsilon G} gHg^{-1}$. Then W is a normal subgroup of G.
40. (**NBHM** (PhD) 2015 Section1 Problem-1.2) Which of the following statements are true?
a. Every group of order 51 is cyclic.
b. Every group of order 151 is cyclic.
c. Every group of order 505 is cyclic
41. (**NBHM** (PhD) 2015 Section1 Problem-1.4) How many elements of order 7 are there in a group of order 28?
42. (**NBHM** (PhD) 2015 Section1 Problem 1.5) Which of the following equations can occur as the class equation of a group of order 10?
a. $10 = 1 + 1 + 1 + 2 + 5$
b. $10 = 1 + 2 + 3 + 4$
c. $10 = 1 + 1 + \cdots + 1(10 times)$

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