Problems List For The Month

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1.Let a, b, c be three distinct integers ,and let P be a polynomial with integer coefficients. Show that in this case the conditions

$$P(a) = b, P(b) = c, P(c) = a$$

Cannot be satisfied simultaneously .

2.Suppose that $P(x)$ is a polynomial of degree n such that

$$P(k) = \frac{k}{k+1} \text{ for } k + 0, 1, 2, \cdots n \text{ .}$$

Find the value of $P(n + 1)$.

3.Given a monic polynomial $f(x)$ of degree n over Z and $k, p, \in N$, prove that if none of the numbers $f(k), f(k + 1), f(k + 2), \cdots f(k + p)$ is divisible by $p + 1$ then $f(x) = 0$ has no rational solution .

4.Show that the polynomial $x^{2n} - 2x^{2n-1} + 3x^{2n-2} - \cdots - 2nx + 2n + 1$ has no real roots .

5.The polynomial $ax^3 + bx^2 + cx + d$ has integral coefficients a, b, c, d with ad odd and bc even .Show that at least one zero of the polynomial is irrational .

6.Let a, b be integers .Then show that the polynomial $(x - a)^2(x - b)^2 + 1$ is not the product of two polynomial with integral coefficients .

7.Let $f(x) = ax^2 + bx + c$. Suppose $f(x) = x$ has no real roots .Show that the equation $f(f(x)) = x$ has no real solutions .

8. Let $f(x)$ be a monic polynomial with integral coefficients. If there are four different integers a, b, c, d, so that $f(a) = f(b) = f(c) = f(d) = 5$, then show that there is no integer ,so that $f(k) = 8$.

9.If $a_1, \cdots, a_n \in Z$ are distinct ,then $(x - a_1) \cdots (x - a_n) - 1$ is irreducible .

$a_1, \cdots, a_{n-1}$ has n real roots . Prove that $P(2) \geq 3^n$.

12.A polynomial $f(x)$ over Z has no integer zero if $f(0)$ and $f(1)$ are both odd.

13.I f $f(x, y, z)$ is symmetric and $x - y | f(x, y, z)$, then $(x-y)^2(y-z)^2(z-x)^2 | f(x, y, z)$.

14.Three integers $p, p+2, p+6$ which are all prime are called a *prime −triplet* .Find fives sets of prime −triplets .

15.If p and $p^2 + 8$ are both prime numbers ,prove that $p^3 + 4$ is also prime .

16.Given a positive integer $k > 1$,show that there are infinitely many integers n for which $\tau(n) = k$, but at most finitely many n with $\sigma(n) = k$.

17.If n and $n+2$ are a pair of *twin primes* ,establish that

$\sigma(n+2) = \sigma(n) + 2$; this also holds for $n = 434$ and 8575 .

18.Prove that *Goldbach's Conjectcture* implies that for each even integer $2n$ there exist integers $n-1$ and n_2 with $\sigma(n_1) + \sigma(n_2) = 2n$.

19.Show that there are infinitely many integers n for which $\phi(n)$ is a perfect square .

20.Prove that the equation $\phi(n) = 2p$, where p is a prime number and $2p+1$ is composite , is not solvable .

21. Use Euler's theorem to confirm that , for any integer $n \geq 0$,

$$51 | 10^{32n+9} - 7.$$

22.Prove that $gcd(2^{15} - 2^3$ divides $a^{15} - a^3$ for any integer a .

23.Prove that every prime other than 2 or 5 divides infinitely many of the integers , 1, 11,111,1111, $\cdots$.

24.Show that if $gcd(a, n) = gcd(a-1, n) = 1$,then

$1 + a + a^2 + \cdots + a^{\phi(n)-1} \equiv 0 \; mod(n)$.

25.For any integer $n \geq 1$,establish the inequality $\tau(n) \leq 2\sqrt{n}$.

26. Let z_1, z_2, z_3 be complex number such that

$$z_1 + z_2 + z_3 = z_1z_2 + z_2z_3 + z_3z_1 = 0.$$

Prove that $|z_1| = |z_2| = |z_3|$.

27. Prove that for all complex number z with $|z| = 1$ the following inequalities hold :

$$\sqrt{2} \leq |1 - z| + |1 + z^2| \leq 4 .$$

2) $z_1 + z_2 + z_3 \neq 0$;

3)$z_1^2 + z_2^2 + z_3^2 = 0$.

Prove that for all integers $n \geq 2$,

$$|z_1^n + z_2^n + z_3^n| \in 0, 1, 2, 3 .$$

29. Let z be a complex number such that $|z| = 1$ and both $Re(z)$ and $Im(z)$ are rational numbers .Prove that $|z^{2n} - 1|$ is rational for all integers $n \geq 1$.

30. Let a, b, c be nonzero complex numbers . Prove that the equation

$$az^3 + bz^2 + \overline{b}z + \overline{a} = 0$$

has at least one root with absolute value equal to 1 .

31. Prove that

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}.$$

32. Let $n \geq 4$ and let $a_1, a_2, \ldots \ldots, a_n$ be the coordinates of the vertices of a regular polygon . Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = a_1a_3 + a_2a_4 + \cdots + a_na_2.$$

33. (Telescoping product.) Prove that

$$\frac{1}{15} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100} < \frac{1}{10}.$$

34.(Telescoping series.) Let $Q_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$. Then , for $n \geq 3$,

$$\frac{19}{12} - \frac{1}{n+1} < Q_n \frac{7}{4} - \frac{1}{n}.$$

35. The *Fibonacci sequence* is defined by $a_1 = a_2 = 1, a_{n+2} = a_n + a_{n+1}$. Prove that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \frac{8}{2^6} + \cdots + \frac{a_n}{2^n} < 2.$$

36. Let $0 < a \leq b \leq c \leq d$ .Then $a^b b^c c^d d^a \geq b^a c^b d^c a^d$.

37. Let ab and $a + b$ have the same sign , then

$$(a+b)(a^4+b^4) \geq (a^2+b^2)(a^3+b^3).$$

38. $a, b, c > 0, a + b + c = 1 \Rightarrow (a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2 \geq \frac{100}{3}$.

39. Let $x_1, \ldots \ldots, x_n$ be positive with $x_1 \cdot x_2 \cdot x_3 \cdot sx_n = 1$.Prove that

$$x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1} \geq \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}.$$