Notation and Conventions

$\mathbb{Z}$ = set of integers
$\mathbb{N}$ = set of natural numbers
$\mathbb{Q}$ = set of rational numbers
$\mathbb{R}$ = set of real numbers
$\mathbb{C}$ = set of complex numbers
$\mathbb{R}^n$ = Euclidean space of dimension n

For a natural number n, the product of all the natural numbers from 1 upto n is denoted by $n!$

$[a, b] = \{x \in \mathbb{R} : \ a \leq x \leq b\}$ for real numbers a and b with $a < b$.

$(a, b) = \{x \in \mathbb{R} : \ a < x < b\}$ for real numbers a and b with $a < b$.

For a differentiable function $f : \mathbb{R} \to \mathbb{R}$, f' denotes its derivative.

For any natural number n, $\mathbb{Z}/n\mathbb{Z}$ denotes the ring of integers modulo n.

Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and metric.

PART A

1. Consider the sequence $\{x_n\}$ defined by $x_n = \frac{[nx]}{n}$ for $x \in \mathbb{R}$ where $[\cdot]$ denotes the integer part. Then $\{x_n\}$

   (a) converges to x.

   (b) converges but not to x.

   (c) does not converge

   (d) oscillates

2. $\lim_{x\to 0} x \sin(1/x^2)$ equals

   (a) 1.

   (b) 0.

   (c) ∞.

   (d) oscillates

3. Let A be a 5×5 matrix with real entries, then A has

   (a) an eigenvalue which is purely imaginary.

   (b) at least one real eigenvalue.

   (c) at least two eigenvalues which are not real.

   (d) at least 2 distinct real eigenvalues.

4. The groups Z_9 and $Z_3 \times Z_3$ are

   (a) isomorphic

   (b) abelian

   (c) non abelian

   (d) cyclic

5. The differential equation

$$\frac{dy}{dx} = y^{\frac{1}{3}}, y(0) = 0$$

   has

   (a) a unique solution

   (b) no nontrivial solution

   (c) finite number of solutions.

   (d) infinite number of solutions.

6. The function $f_n(x) = n \sin(x/n)$

   (a) does not converge for any x as $n \to \infty$.
   (b) converges to the constant function 1 as $n \to \infty$.
   (c) converges to the function x as $n \to \infty$.
   (d) does not converge for all x as $n \to \infty$.

7. The equation $x^{22} \equiv 2 \mod 23$ has

   (a) no solutions.
   (b) 23 solutions.
   (c) exactly one solution.
   (d) 22 solutions.

8. The sum of the squares of the roots of the cubic equation $x^3 - 4x^2 + 6x + 1$ is

   (a) 0.
   (b) 4.
   (c) 16.
   (d) none of the above.

9. The function $f(x)$ defined by

$$f(x) = \begin{cases} ax + b & x \geq 1, \\ x^2 + 3x + 3 & x \leq 1 \end{cases}$$

   is differentiable

   (a) for a unique value of a and infinitely many values of b.
   (b) for a unique value of b and infinitely many values of a.
   (c) for infinitely many values of a and b.
   (d) none of the above.

10. Let $m \leq n$ be natural numbers. The number of injective maps from a set of cardinality m to a set of cardinality n is

   (a) $m!$
   (b) $n!$
   (c) $(n - m)!$
   (d) none of the above.

11. For any real number c, the polynomial $x^3 + x + c$ has exactly one real root.

12.
$$e^{\sqrt{2}} > 3.$$

13. A is 3×4-matrix of rank 3. Then the system of equations,
$$Ax = b$$
has exactly one solution.

14. $\log x$ is uniformly continuous on $(\frac{1}{2}, \infty)$.

15. If $A,\ B$ are closed subsets of $[0, \infty)$, then
$$A + B = \{x + y \mid x \in A,\ y \in B\}$$
is closed in $[0, \infty)$.

16. The polynomial $x^4 + 7x^3 - 13x^2 + 11x$ has exactly one real root.

17. The value of the infinite product
$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$$
is 1.

18. Consider the map T from the vector space of polynomials of degree at most 5 over the reals to $\mathbb{R} \times \mathbb{R}$, given by sending a polynomial P to the pair $(P(3), P'(3))$ where P' is the derivative of P. Then the dimension of the kernel is 3.

19. The derivative of the function

$$\int_0^{\sqrt{x}} e^{-t^2} dt$$

at $x = 1$ is e^{-1}.

20. The equation $63x + 70y + 15z = 2010$ has an integral solution.

21. Any continuous function from the open unit interval $(0, 1)$ to itself has a fixed point.

22. There exists a group with a proper subgroup isomorphic to itself.

23. The space of solutions of infinitely differentiable functions satisfying the equation

$$y" + y = 0$$

is infinite dimensional.

24. The series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

diverges.

25. The function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous anywhere on the real line.

PART B

1. Let A be a 2×2-matrix with complex entries. The number of 2×2-matrices A with complex entries satisfying the equation $A^3 = A$ is infinite.

2. In the ring $\mathbb{Z}/8\mathbb{Z}$, the equation $x^2 = 1$ has exactly 2 solutions.

3. There are n homomorphisms from the group $\mathbb{Z}/n\mathbb{Z}$ to the additive group of rationals $\mathbb{Q}$.

4. A bounded continuous function on $\mathbb{R}$ is uniformly continuous.

5. The symmetric group S_5 consisting of permutations on 5 symbols has an element of order 6.

6. Suppose $f_n(x)$ is a sequence of continuous functions on the closed interval $[0, 1]$ converging to 0 pointwise. Then the integral
$$\int_0^1 f_n(x)dx$$
converges to 0.

7. There is a non-trivial group homomorphism from S_3 to $\mathbb{Z}/3\mathbb{Z}$.

8. If A and B are 3×3 matrices and A is invertible, then there exists an integer n such that $A + nB$ is invertible.

9. Let P be a degree 3 polynomial with complex coefficients such that the constant term is 2010. Then P has a root α with $|\alpha| > 10$.

10. Suppose a box contains three cards, one with both sides white, one with both sides black, and one with one side white and the other side black. If you pick a card at random, and the side facing you is white, then the probability that the other side is white is $1/2$.

11. There exists a set $A \subset \{1, 2, \cdots, 100\}$ with 65 elements, such that 65 cannot be expressed as a sum of two elements in A.

12. Let S be a finite subset of $\mathbb{R}^3$ such that any three elements in S span a two dimensional subspace. Then S spans a two dimensional space.

13. Any non-singular $k \times k$-matrix with real entries can be made singular by changing exactly one entry.

14. Let f be a continuous integrable function of $\mathbb{R}$ such that either $f(x) > 0$ or $f(x) + f(x+1) > 0$ for all $x \in \mathbb{R}$. Then $\int_{-\infty}^{\infty} f(x)dx > 0$.

15. A gardener throws 18 seeds onto an equilateral triangle shaped plot of land with sides of length one metre. Then at least two seeds are within a distance of 25 centimetres.