MADHAVA MATHEMATICS COMPETITION, December 2015
Solutions and Scheme of Marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let $A(t)$ denote the area bounded by the curve $y = e^{-|x|}$, the $X-$ axis and the straight lines $x = -t, x = t$, then $\lim_{t\to\infty} A(t)$ is
A) 2 B) 1 C) 1/2 D) e.
Solution: (A)
As $f(x) = e^{-|x|}$ is an even function, $A(t) = 2\int_{-t}^{0}\int_{0}^{e^x} 1\, dydx = 2\int_{-t}^{0} e^x dx = 2(e^0 - e^{-t})$
$\to 2$ as $t \to \infty$. **OR** $A(t) = 2\int_0^t e^x\, dx = -2(e^{-t} - 1) \to 2$ as $t \to \infty$.

2. How many triples of real numbers (x, y, z) are common solutions of the equations
$x + y = 2, \quad xy - z^2 = 1$?
A) 0 B) 1 C) 2 D) infinitely many.
Solution: (B)
$xy = 1+z^2 \geq 1$ so that $-4xy \leq -4$. Hence $(x-y)^2 = (x+y)^2-4xy = 4-4xy \leq 4-4 = 0$. So $x = y$. Thus the only solution is $x = 1, y = 1, z = 0$.

3. For non-negative integers x, y the function $f(x, y)$ satisfies the relations $f(x, 0) = x$ and $f(x, y + 1) = f(f(x, y), y)$. Then which of the following is the largest?
A) $f(10, 15)$ B) $f(12, 13)$ C) $f(13, 12)$ D) $f(14, 11)$.
Solution: (D)
$f(x, 1) = f(f(x, 0), 0) = f(x, 0) = x$. Inductively $f(x, y) = x$ for all integers $y \geq 0$.

4. Suppose p, q, r, s are $1, 2, 3, 4$ in some order. Let $x = \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s}}}}$.
We choose p, q, r, s so that x is as large as possible, then s is
A) 1 B) 2 C) 3 D) 4.
Solution: (C)
For x to be the largest, p, q, r, s should be $\min\{1, 2, 3, 4\}$, $\max\{1, 2, 3, 4\}$, $\min\{2, 3\}$, $\max\{2, 3\}$ respectively. So $s = 3$.

5. Let $f(x) = \begin{cases} 3x + x^2 & \text{if } x < 0 \\ x^3 + x^2 & \text{if } x \geq 0. \end{cases}$ Then $f''(0)$ is
A) 0 B) 2 C) 3 D) None of these.
Solution: (D)
$f'_-(x) = 3 + 2x$, for $x < 0$ and $f'_+(x) = 3x^2 + 2x$, for $x \geq 0$. So $f'(0)$ does not exist because $f'_-(0) = 3 \neq 0 = f'_+(0)$.

6. There are 8 teams in pro-kabaddi league. Each team plays against every other exactly once. Suppose every game results in a win, that is, there is no draw. Let $w_1, w_2, \cdots, w_8$ be number of wins and $l_1, l_2, \cdots, l_8$ be number of loses by teams $T_1, T_2, \cdots, T_8$, then
A) $w_1^2 + \cdots + w_8^2 = 49 + (l_1^2 + \cdots + l_8^2)$. B) $w_1^2 + \cdots + w_8^2 = l_1^2 + \cdots + l_8^2$.
C) $w_1^2 + \cdots + w_8^2 = 49 - (l_1^2 + \cdots + l_8^2)$. D) None of these.
Solution: (B)
Note that $w_i + l_i = 7$ for all i and $\sum w_i - \sum l_i = \sum(w_i - l_i) = 0$. Then $\sum w_i^2 - \sum l_i^2 = \sum(w_i + l_i)(w_i + l_i) = 7\sum(w_i - l_i) = 0$.

7. The remainder when $m+n$ is divided by 12 is 8, and the remainder when $m-n$ is divided by 12 is 6. If $m > n$, then the remainder when mn divided by 6 is
A) 1 B) 2 C) 3 D) 4 .
Solution: (A)
Note that $m+n \equiv 8(\text{mod } 12)$ and $m-n \equiv 6(\text{mod } 12)$. Adding these congruences, we get, $2m \equiv 2(\text{mod } 12)$. This implies $m \equiv 1(\text{mod } 6)$. Similarly by subtracting, we get, $n \equiv 1(\text{mod } 6)$. Thus $mn \equiv 1(\text{mod } 6)$.

8. Let $A = \begin{pmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \ddots & \vdots & \\ (n-1)n+1 & (n-1)n+2 & \dots & n^2 \end{pmatrix}$. Select any entry and call it x_1. Delete row and column containing x_1 to get an $(n-1) \times (n-1)$ matrix. Then select any entry from the remaining entries and call it x_2. Delete row and column containing x_2 to get $(n-2) \times (n-2)$ matrix. Perform n such steps. Then $x_1 + x_2 + \cdots + x_n$ is
A) n B) $\frac{n(n+1)}{2}$ C) $\frac{n(n^2+1)}{2}$ D) None of these.
Solution: (C)
For $n = 4$, $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1+4 & 2+4 & 3+4 & 4+4 \\ 1+8 & 2+8 & 3+8 & 4+8 \\ 1+12 & 2+12 & 3+12 & 4+12 \end{pmatrix}$. Note that $S = x_1 + x_2 + x_3 + x_4 = a + (b+4) + (c+8) + (d+12)$ where a, b, c, d is a permutation of $1, 2, 3, 4$. So $S = a+b+c+d+24 = 10+24 = 34$. Thus S is the same for all stated choices of $x_1, \dots, x_n$. Hence taking x_i's as the main diagonal elements,

$$\begin{aligned} S &= 1 + (n+2) + (2n+3) + \ldots + [n(n-1)+n] \\ &= [n + 2n + \ldots + n(n-1)] + [1 + 2 + \ldots + n] \\ &= \frac{n(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2}. \end{aligned}$$

9. The maximum of the areas of the rectangles inscribed in the region bounded by the curve $y = 3 - x^2$ and $X-$axis is
A) 4 B) 1 C) 3 D) 2.
Solution: (A)
By symmetry, let the base of the rectangle be segment with ends $-x, x$ and height y. Then area $A(x) = 2xy = 2x(3-x^2) = 6x - 2x^3$ and $A'(x) = 6 - 6x^2$, $A'(x) = -12x$. So $A'(x) = 0$ at $x^2 = 1$ i.e. $x = 1$; and $A''(1) = -12 < 0$. So $A(x)$ is maximum at $x = 1$ with maximum value $A(1) = 4$.

10. How many factors of $2^5 3^6 5^2$ are perfect squares?
A) 24 B) 20 C) 30 D) 36.
Solution: (A)
Factors that are perfect squares will be of the form $d = 2^a 3^b 5^c$ where $a = 0, 2$ or 4, $b = 0, 2, 4$ or 6, and $c = 0$ or 2. Thus there are $3 \times 4 \times 2$ possible divisors that are perfect squares.

Part II

N.B. Each question in Part II carries 6 marks.

1. How many 15−digit palindromes are there in each of which the product of the non-zero digits is 36 and the sum of the digits is equal to 15? (A string of digits is called a palindrome if it reads the same forwards and backwards. For example 04340, 6411146.)
Solution: The first 7 digits completely determine the number. Since the sum of digits is 15, the 8^{th} digit is odd and is a factor of 36. Note that the product of all non-zero digits (except the digit in the 8^{th} place) is a square using the definition of palindrome. Hence

the 8^{th} digit cannot be 3 because the product in that case is 12. So the 8^{th} digit is either 1 or 9. **[2]**

Case 1: If the 8^{th} digit is 1 then the digits in first seven places can either be a permutation of 1,1,2,3,0,0,0 or 1,6,0,0,0,0,0 because these are the only possibilities with sum 7 and product 6.

Number of permutations of 1,1,2,3,0,0,0 is $\frac{7!}{2!3!}$.

Number of permutations of 1,6,0,0,0,0,0 is $\frac{7!}{5!}$. **[2]**

Case 2: If the 8^{th} digit is 9 then the digits in first seven places will be a permutation of 1,2,0,0,0,0,0 because this is the only possibility with sum 3 and product 2..

Number of permutations of 1,2,0,0,0,0,0 is $\frac{7!}{5!}$.

Thus the number of required 15 digit palindromes with product of nonzero digits 36 and sum of digits 15 is $\frac{7!}{2!3!}+\frac{7!}{5!}+\frac{7!}{5!}$. **[2]**

2. Let H be a finite set of distinct positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3. Find two different such sets with sum of the reciprocals equal to 2.5.

   Solution: The given condition implies that every $n \in H$, n is of the form $n = 2^\alpha 3^\beta$, $\alpha, \beta \geq 0$. Since H is finite, $\exists k \in \mathbb{N}$ such that $\alpha \leq k$, $\beta \leq k$ for each $n \in H$. This implies

$$\sum_{n\in H}\frac{1}{n} \leq 1 + \sum_{i=1}^{k}\frac{1}{2^i} + \sum_{j=1}^{k}\frac{1}{3^j} + \sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{2^i3^j}$$

$$= 1 + \sum_{i=1}^{k}\frac{1}{2^i} + \sum_{j=1}^{k}\frac{1}{3^j} + \left(\sum_{i=1}^{k}\frac{1}{2^i}\right)\left(\sum_{j=1}^{k}\frac{1}{3^j}\right)$$

$$= \left(1+\frac{1}{2}+\cdots+\frac{1}{2^k}\right)\left(1+\frac{1}{3}+\cdots+\frac{1}{3^k}\right) \quad \textbf{[3]}$$

$$= \left(\frac{1-\frac{1}{2^{k+1}}}{1-1/2}\right)\left(\frac{1-\frac{1}{3^{k+1}}}{1-1/3}\right) < \left(\frac{1}{1/2}\right)\left(\frac{1}{2/3}\right) = 2\left(\frac{3}{2}\right) = 3. \quad \textbf{[1]}$$

   Let $H = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Then $\sum_{n\in H}\frac{1}{n} = 2.5$. **[1]**

   Let $H = \{1, 2, 3, 4, 6, 8, 12, 36, 72\}$. Then $\sum_{n\in H}\frac{1}{n} = 2.5$. **[1]**

   Any other correct choices for H, also carries one mark each.

3. Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. Find the sum of all entries of A^{2015}.

   Solution: Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. This implies

$$A\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}. \quad \textbf{[3]}$$

   By repeated use of this, we get

$$A^{2015}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}. \quad \textbf{[2]}$$

   So each row sum of A^{2015} is equal to one. Hence the sum of all entries of A^{2015} is n. **[1]**

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f'(x) > f(x)$ for all $x \in \mathbb{R}$.

Prove that $f(x) > 0$ for all $x > 0$.

Solution: By data, $f'(0) > f(0) = 0$. So $\lim_{x\to 0+} \frac{f(x)}{x} > 0$. Hence $\exists\ \delta > 0$, such that $f(x) > 0$ for all $x \in (0, \delta)$. **[1]**

Now if there exists $x_0\ > 0$ such that $f(x_0) \le 0$, then by intermediate value property, there exists $x_1\ \ge \delta$ such that $f(x_1) = 0$. Let $c = \inf A$, $\ A = \{x \mid x > 0\ ,\ f(x) = 0\}$. **[1]**

Clearly, as f is continuous and c is $\inf A$, $f(x) \ge 0$, for $x \in [0, c]$. So $c \ge \delta$. By the property of infimum, there exists a sequence $\{x_n\}$ of points which converges to c, and $x_n > 0$ and $f(x_n) = 0$. By continuity, $f(c) = \lim_{n\to\infty} f(x_n) = 0$. **[1]**

This implies that in $(0, c)$, $f(x) > 0$ and $f(0) = f(c) = 0$. Hence by Rolle's theorem, $f'(b) = 0$ for some $b,\ 0 < b < c$. **[2]**

But then $f(b) < f'(b) = 0$ which is a contradiction. Hence $f(x) > 0$ for all $x > 0$. **[1]**

Second method. As before, $\exists\ \delta > 0$, such that $f(x) > 0$ for all $x \in (0, \delta)$. **[1]**

Let $S = \{x \mid f(t) > 0 \text{ for } t \in (0, x).\}$ Then $\delta \in S$ so that S is non-empty. Let $m = \sup S$. If $m = \infty$, we are done. Let, if possible, $m < \infty$. Now $f(x) > 0$, $x \in (0, m)$. By continuity, $f(m) = \lim_{t\to m-} f(t) \ge 0$. **[2]**

So for all $x \in [0, m]$, $f'(x) > f(x) \ge 0$ so that $f'(x) > 0$. Hence f is strictly increasing on $[0, m]$, in particular, $f(m) > f(m/2) > 0$. Since $f(m) > 0$, by continuity, there exists $\delta_1 > 0$ such that $f(x) > 0$ in $[m, m + \delta_1)$. So, $f(x) > 0$, for $x \in (0, m + \delta_1)$. Thus $m + \delta_1 \in S$, which is a contradiction since $m = \sup S$. Hence $m = \infty$. **[3]**

5. Give an example of a function which is continuous on $[0, 1]$, differentiable on $(0, 1)$ and not differentiable at the end points. Justify.

   Solution: $f(x) = \sqrt{x - x^2}$ for $x \in [0, 1]$. **[3]**

   Then $f'(x)$ exists on $(0, 1)$, $f'(x) = \frac{1 - 2x}{2\sqrt{x - x^2}}$. **[1]**

   But $f'(0) = f'(1) = \infty$. **[2]**

   Note: Any other correct example with justification will carry full marks.

Part III

1. There are some marbles in a bowl. A, B and C take turns removing one or two marbles from the bowl, with A going first, then B, then C, then A again and so on. The player who takes the last marble from the bowl is the loser and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can B and C work together and force A to lose? [12]

   Solution: We claim that B and C can force A to lose for all N except
   N = 2; 3; 4; 7; or 8.
   At N = 2, A leaves 1.
   At N = 3 or 4, A leaves 2.
   At N = 7 or 8, A leaves 6 after which B and C must leave 2, 3 or 4.
   For N = 5 or 6, regardless of what A takes, B and C can work it so that when A's turn arrives there is only one marble left.
   For N = 9 or 10, A must leave 7, 8 or 9 from which B and C can force 5 or 6. **[6]**
   For N = $4k$ where $k > 2$, A must leave either $4k - 1$ or $4k - 2$ from which B and C can force $4(k - 1) + 1$ or $4(k - 2) + 2$.
   For N = $4k+1$, A must leave either $4k$ or $4k-1$ from which B and C can force $4(k-1)+2$ or $4(k - 1) + 1$.
   For N = $4k+2$, A must leave either $4k+1$ or $4k$ from which B and C can force $4(k-1)+2$ or $4(k - 1) + 1$.
   For N = $4k + 3$, A must leave either $4k + 2$ or $4k + 1$ from which B and C can force $4(k - 1) + 2$.

In all cases for N ≥ 11, A will always be faced with a new value of the form $4t+1$ or $4t+2$ on his next turn eventually forcing him to N $= 5$ or 6 and a loss. **[6]**

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(0)$ exists. Suppose $\alpha_n \neq \beta_n, \forall n \in \mathbb{N}$ and both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. Define $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$.
Prove that $\lim_{n\to\infty} D_n = f'(0)$ under EACH of the following conditions:

   (a) $\alpha_n < 0 < \beta_n, \ \forall n \in \mathbb{N}$.

   (b) $0 < \alpha_n < \beta_n$ and $\dfrac{\beta_n}{\beta_n - \alpha_n} \leq M, \quad \forall n \in \mathbb{N}$, for some $M > 0$.

   (c) $f'(x)$ exists and is continuous for all $x \in (-1, 1)$. [13]

Solution: Let $\epsilon > 0$ be given. Given that $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x}$ exists.

(a) Given that $\alpha_n < 0 < \beta_n, \ \forall n \in \mathbb{N}$. Since $\alpha_n \to 0$ and $\beta_n \to 0$, we have

$f'(0) = \lim_{n\to\infty} \dfrac{f(\alpha_n) - f(0)}{\alpha_n}$ and $f'(0) = \lim_{n\to\infty} \dfrac{f(\beta_n) - f(0)}{\beta_n}$.

There exist $n_1, n_2 \in \mathbb{N}$ such that $|f(\alpha_n) - f(0) - \alpha_n f'(0)| < |\alpha_n|\epsilon = -\alpha_n\epsilon, \quad \forall n \geq n_1$ and

$|f(\beta_n) - f(0) - \beta_n f'(0)| < \ |\beta_n|\epsilon = \beta_n\epsilon, \quad \forall n \geq n_2$.

Let $n_0 = \max\{n_1, n_2\}$. Then $\forall n \geq n_0$, we get

$|f(\beta_n) - f(\alpha_n) - (\beta_n - \alpha_n)f'(0)|$

$\leq |f(\beta_n) - f(0) - \beta_n f'(0)| + |f(\alpha_n) - f(0) - \alpha_n f'(0)| < \ (\beta_n - \alpha_n)\epsilon.$

Thus $|\dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - f'(0)| < \ \epsilon, \quad \forall n \geq n_0$.

Hence $\lim_{n\to\infty} D_n = f'(0)$. **[4]**

(b) Given that $0 < \alpha_n < \beta_n$ and $\dfrac{\beta_n}{\beta_n - \alpha_n} \leq M, \quad \forall n \in \mathbb{N}$, for some $M > 0$.
Since $\alpha_n < \beta_n$, observe that $\dfrac{\alpha_n}{\beta_n - \alpha_n} \leq M, \quad \forall n \in \mathbb{N}$.
Similar to part (a), there exist $n_1, n_2 \in \mathbb{N}$ such that $|f(\alpha_n) - f(0) - \alpha_n f'(0)| < \ |\alpha_n|\epsilon = \alpha_n\epsilon, \quad \forall n \geq n_1$ and $|f(\beta_n) - f(0) - \beta_n f'(0)| < \ |\beta_n|\epsilon = \beta_n\epsilon, \quad \forall n \geq n_2$.
Let $n_0 = \max\{n_1, n_2\}$. Then $\forall n \geq n_0$, we get

$$|\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - f'(0)|$$
$$= |\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - (\frac{\beta_n - \alpha_n}{\beta_n - \alpha_n})f'(0)|$$
$$= |\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - \frac{\beta_n f'(0)}{\beta_n - \alpha_n} + \frac{\alpha_n f'(0)}{\beta_n - \alpha_n}|$$
$$= |(\frac{f(\beta_n) - f(0) - \beta_n f'(0)}{\beta_n - \alpha_n}) - (\frac{f(\alpha_n) - f(0) - \alpha_n f'(0)}{\beta_n - \alpha_n})|$$
$$\leq |\frac{f(\beta_n) - f(0) - \beta_n f'(0)}{\beta_n - \alpha_n}| + |\frac{f(\alpha_n) - f(0) - \alpha_n f'(0)}{\beta_n - \alpha_n}|$$
$$< \ (\frac{\beta_n}{\beta_n - \alpha_n})\epsilon + (\frac{\alpha_n}{\beta_n - \alpha_n})\epsilon \leq 2M\epsilon.$$

Hence $\lim_{n\to\infty} D_n = f'(0)$. **[5]**

(c) Given that $f'(x)$ exists and is continuous for all $x \in (-1, 1)$.
By Lagrange's Mean Value Theorem, for every positive integer n, there exists c_n between

α_n and β_n such that
$D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(c_n)$.
Since $\alpha_n \to 0$ and $\beta_n \to 0$, $c_n \to 0$. It is given that $f'(x)$ is continuous.
Therefore $\lim_{n\to\infty} f'(c_n) = f'(0)$.
Hence $\lim_{n\to\infty} D_n = f'(0)$. **[4]**

Second method for (a), (b). Let $g(x) = f(x) - f(0) - xf'(0)$ on $\mathbb{R}$. Then $g(0) = 0$ and $g'(0) = \lim_{x\to 0} \dfrac{g(x)}{x} = \lim_{x\to 0} \left[\dfrac{f(x) - f(0)}{x} - f'(0)\right] = 0$. Let $E_n = \dfrac{g(\beta_n) - g(\alpha_n)}{\beta_n - \alpha_n}$. Then $E_n = D_n - f'(0)$.

(a) Let $\alpha_n < 0 < \beta_n$. Then $0 < -\alpha_n < \beta_n - \alpha_n$. Hence $0 < \dfrac{-\alpha_n}{\beta_n - \alpha_n} < 1$. Also, here $\beta_n - \alpha_n > \beta_n$ so that $0 < \dfrac{\beta_n}{\beta_n - \alpha_n} < 1$. Hence

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{g(\beta_n)}{\beta_n}\left(\frac{\beta_n}{\beta_n - \alpha_n}\right) + \lim_{n\to\infty} \frac{g(\alpha_n)}{\alpha_n}\left(\frac{-\alpha_n}{\beta_n - \alpha_n}\right) = 0,$$

as the sequences in brackets are both bounded and $g'(0) = 0$. **[4]**

(b) Let $0 < \alpha_n < \beta_n$ and $0 < \dfrac{\beta_n}{\beta_n - \alpha_n} < M$. Then $\beta_n - \alpha_n > 0$ and so

$0 < \dfrac{\alpha_n}{\beta_n - \alpha_n} < \dfrac{\beta_n}{\beta_n - \alpha_n} < M$. Hence

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{g(\beta_n)}{\beta_n}\left(\frac{\beta_n}{\beta_n - \alpha_n}\right) - \lim_{n\to\infty} \frac{g(\alpha_n)}{\alpha_n}\left(\frac{\alpha_n}{\beta_n - \alpha_n}\right) = 0, \text{ as in (a).}$$ **[5]**

3. Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Then draw a tangent at P_2 and so on. Show that the ratio $\dfrac{A(\triangle P_n P_{n+1} P_{n+2})}{A(\triangle P_{n+1} P_{n+2} P_{n+3})}$ is constant. [12]

Solution: Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Recursively P_{n+1} is defined. We now try to calculate P_2 in terms of P_1. Tangent at P_1 is given by $y - y_1 = 5x_1^4(x - x_1)$ i.e. $y = 5x_1^4 x - 4x_1^5$. This cuts the curve $y = x^5$ at $x = x_2$, $x_2 \neq x_1$. Hence

$$x_2^5 - 5x_1^4 x_2 + 4x_1^5 = 0.$$

This is a *homogeneous* equation in x_1, x_2. So put $x_2 = kx_1$, $k \neq 1$, then $x_1^5 k^5 - 5kx_1^5 + 4x_1^5 = 0$. This implies $k^5 - 5k + 4 = 0$ i.e. $(k-1)^2(k^3 + 2k^2 + 3k + 4) = 0$. Since $k \neq 1$, $k^3 + 2k^2 + 3k + 4 = 0$. Observe that this cubic equation has one real and two complex roots. ($g'(x) = 3k^2 + 4k + 3 \neq 0$.) The real root k must be negative. [5]

If $x_3 = \ell x_2$, then by similar argument, we see that ℓ is again the above negative root k of $k^3 + 2k^2 + 3k + 4 = 0$. Thus $x_3 = kx_2 = k^2 x_1$. Hence by induction, $x_{n+1} = k^n x_1$ for all $n \geq 1$. [2]
We now calculate $A(\triangle P_n P_{n+1} P_{n+2})$

$$A(\triangle P_n P_{n+1} P_{n+2}) = \frac{1}{2} \det \begin{pmatrix} x_n & x_n^5 & 1 \\ x_{n+1} & x_{n+1}^5 & 1 \\ x_{n+2} & x_{n+2}^5 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \det \begin{pmatrix} k^{n-1}x_1 & k^{5n-5}x_1^5 & 1 \\ k^n x_1 & k^{5n}x_1^5 & 1 \\ k^{n+1}x_1 & k^{5n+5}x_1^5 & 1 \end{pmatrix}$$

$$= \frac{k^{n-1}x_1k^{5n-5}x_1^5}{2}\det\begin{pmatrix}1 & 1 & 1\\ k & k^5 & 1\\ k^2 & k^{10} & 1\end{pmatrix}$$

$$= k^{6n-6}x_1^6D, \text{ where } D = \frac{1}{2}\det\begin{pmatrix}1 & 1 & 1\\ k & k^5 & 1\\ k^2 & k^{10} & 1\end{pmatrix} \neq 0 \text{ since } k \neq -1.$$

Then $A(\triangle P_{n+1}P_{n+2}P_{n+3}) = k^{6n}x_1^6D$.

Hence the ratio $\dfrac{A(\triangle P_nP_{n+1}P_{n+2})}{A(\triangle P_{n+1}P_{n+2}P_{n+3})} = \dfrac{1}{k^6}$ is constant. [5]

4. Let $p(x)$ be a polynomial with positive integer coefficients. You can ask the question: What is $p(n)$ for any positive integer n? What is the minimum number of questions to be asked to determine $p(x)$ completely? Justify. [13]

Solution: The minimum number of questions needed is 2. For this, let $p(x)$ be a polynomial with positive integer coefficients say, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$. We can ask the question: what is $p(1)$? Let $p(1) = N$. [3]

Here $N = a_0 + a_1 + a_2 + \cdots + a_k > a_i,\ \forall i$ and N is a known number.

Also, what is $p(N)$? So $p(N) = a_0 + a_1N + a_2N^2 + \cdots + a_kN^k$ is a known number. [4]

Now express $p(N)$ to base N, then i^{th} digit gives $a_i,\ \forall i$ because $a_i < N,\ \forall i$. Thus $p(x)$ is determined. [6]
Note that asking only one question i.e. asking for the value $p(n)$ for a particular choice of n, is not sufficient to determine the polynomial $p(x)$.

Example. Suppose $p(1) = 9$ and $p(9) = 193$. Now we express 193 to base 9 :

$$193 = 21(9) + 4, \quad 21 = 2(9) + 3, \quad 2 = 0(9) + 2.$$

So the remainders are, starting with the last, $2, 3, 4$. So $193 = 2(9^2)+3(9)+4(9^0) = (234)_9$. So $a_2 = 2, a_1 = 3, a_0 = 4$ and $p(x) = 4 + 3x + 2x^2$.