MADHAVA MATHEMATICS COMPETITION, JANUARY 5, 2014

N.B. Each question in Part I carries 2 marks. [20]

1. If $x^3 - x + 1 = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3$, then (a_0, a_1, a_2, a_3) equals

   A. $(1, -1, 0, 1)$ **B.** $(7, 6, 10, 1)$ **C.** $(7, 11, 12, 6)$ **D.** $(7, 11, 6, 1)$

   Solution: (D)

   Using Taylor series expansion for $f(x) = x^3 - x + 1$ about $x = 2$, we get $a_r = \dfrac{f^{(r)}(2)}{r!}$. Then $a_0 = 7, a_1 = 11, a_2 = 6, a_3 = 1$.

2. Suppose $f(x)$ and $g(x)$ are real-valued differentiable functions such that $f'(x) \geq g'(x)$ for all x in $[0,1]$. Which of the following is necessarily true?

   A. $f(1) \geq g(1)$ **C.** $f(1) - g(1) \geq f(0) - g(0)$

   B. $f - g$ has no maximum on $[0,1]$ **D.** $f + g$ is a non-decreasing function on $[0,1]$

   Solution: (C)

   The condition $f'(x) - g'(x) \geq 0$ implies that $f(x) - g(x)$ is an increasing function on $[0,1]$. Therefore $f(1) - g(1) \geq f(0) - g(0)$.

3. The equation $x^4 + x^2 - 1 = 0$ has

   A. two positive and two negative roots **C.** one positive, one negative and two non-real roots

   B. all positive roots **D.** no real root

   Solution: (C)

   Put $x^2 = y$. Then the equation becomes $y^2 + y - 1 = 0$. Solving this equation, we get $x^2 = y = \dfrac{-1 \pm \sqrt{5}}{2}$. When $x^2 = \dfrac{-1+\sqrt{5}}{2}$, we get two real values of x, one positive and other negative. But no real x exists such that $x^2 = \dfrac{-1-\sqrt{5}}{2}$. Hence The equation $x^4 + x^2 - 1 = 0$ has one positive, one negative and two non-real roots.

4. Let n be a natural number. Let A and B be $n \times n$ matrices. If A is invertible, then which of the following is necessarily true?

   A. $rank(AB) < rank(B)$ **C.** $rank(AB) = rank(B)$

   B. $rank(AB) > rank(B)$ **D.** $rank(AB) < rank(A)$

   Solution: (C)

   Since A is invertible, it is product of elementary matrices. The matrix AB can be obtained from B by performing elementary row transformations. Therefore $rank(AB) = rank(B)$.

5. Let X be a set and A, B, C be its subsets. Which of the following is necessarily true?

   A. $A - (A - B) = B$ **C.** $A - (B \cup C) = (A - B) \cup (A - C)$

   B. $A - (B \cap C) = (A - B) \cap (A - C)$ **D.** $B - (A - B) = B$

   Solution: (D)

   Note that $A - B = A \cap B'$. Also, $(A - B)' = (A \cap B')' = A' \cup B$.

   Hence $B - (A - B) = B \cap (A - B)' = B \cap (A' \cup B) = (B \cap A') \cup B = B$.

6. For a real number x we let $[x]$ denote the largest integer not exceeding x. For a natural number n, let $a_n = \frac{[n\sqrt{2}]}{n}$. The limit $\lim\limits_{n\to\infty} a_n$

   A. equals 0 **B.** equals $\left[\sqrt{2}\right]$ **C.** equals $\sqrt{2}$ **D.** does not exist

   Solution: (C)

   Note that $(n\sqrt{2}) - 1 \leq \left[n\sqrt{2}\right] \leq n\sqrt{2}$.

   Therefore $\sqrt{2} - \dfrac{1}{n} \leq \dfrac{\left[n\sqrt{2}\right]}{n} \leq \sqrt{2}$. Taking limit as $n \to \infty$, we get The limit $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

7. Let M be a two-digit natural number. Let N be the natural number whose digits are that of M but are in the reverse order. Which of the following CANNOT be the sum of M and N?

A. 181 **B.** 165 **C.** 121 **D.** 154

Solution: (A)

If $M = 10a_1 + a_0$, then $N = 10a_0 + a_1$. Therefore $M + N = 10(a_0 + a_1) + (a_0 + a_1) = (11)(a_0 + a_1)$. Hence $M + N$ is divisible by 11, but 181 is not divisible by 11.

8. The value of $\lim_{x\to 1} \dfrac{\int_1^x e^{t^2}\,dt}{x^2-1}$ is

A. 0 **B.** 1 **C.** e **D.** $e/2$

Solution: (D)

This limit is in the $\dfrac{0}{0}$ form. Then by L'Hospital rule,

$$\lim_{x\to 1} \frac{\int_1^x e^{t^2}\,dt}{x^2-1} = \lim_{x\to 1} \frac{\frac{d}{dx}\int_1^x e^{t^2}\,dt}{2x} = \lim_{x\to 1} \frac{e^{x^2}}{2x} = \frac{e}{2}.$$

9. Let n be any positive integer and $1 \le x_1 < x_2 < \cdots < x_{n+1} \le 2n$, where each x_i is an integer. Which of the following must be true?

(I) There is an i such that x_i is a square of an integer.
(II) There is an i such that $x_{i+1} = x_i + 1$.
(III) There is an i such that x_i is prime.

A. I only **B.** II only **C.** I and II only **D.** II and III only

Solution: (B)

Let $S_{2n} = \{1, 2, \ldots, 2n\}$. If we choose the 6 integers $2, 3, 5, 6, 7, 8$ from S_{10}, then none of them is a square. So (I) is false for $n = 5$. If we choose the 6 integers $1, 4, 6, 8, 9, 10$ from S_{10}, then none of them is a prime. So (III) is false for $n = 5$.

Now (II) is clearly true for $n = 2$. Also, for $n \ge 2$, suppose, if possible, $n + 1$ integers x_i can be chosen such that

$$1 \le x_1 < x_2 < \cdots < x_{n+1} \le 2n,$$

and such that no two of them differ by 1. Then $x_2 \ge x_1 + 2$, $x_3 \ge x_2 + 2$, ... $x_{n+1} \ge x_n + 2$. Adding these, we get $x_{n+1} \ge x_1 + 2n \ge 2n + 1 > 2n$. This contradicts the choice of x_{n+1}. So (II) is true for every n.

10. Two real numbers x and y are chosen uniformly at random from the interval $[0, 1]$. Find the probability that $2x > y$.

A. 1/4 **B.** 1/2 **C.** 2/3 **D.** 3/4

Solution: (D)

The probability that $2x > y$ is the area of the region in the unit square below the line $2x = y$. Thus the required probability is 3/4.

Part II

N.B. Each question in Part II carries 6 marks. [30]

1. Let A be an 8×3 matrix in which every entry is either 1 or -1, and no two rows are identical. Find the rank of A.

Solution: The given conditions imply that the rows of A must be the following triplets, in some order.

$$(1,1,1),\quad (1,1,-1),\quad (1,-1,1),\quad (-1,1,1),$$
$$(1,-1,-1),\quad (-1,1,-1),\quad (-1,-1,1),\quad (-1,-1,-1).$$

To see this, let (a, b, c) be a triplet satisfying the given conditions. Then each of a, b, c can be chosen in one two ways (1 or -1). So there are exactly $2^3 = 8$ such distinct triplets : they are listed above. So they are the rows of A, in some order, as A has size 8×3. [3]
Also, the 3×3 submatrix of A having the triplets $(1, 1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$ as rows in some order is non-singular. So the rank of A is 3. [3]

2. Find all pairs (x, y) of integers such that $y^2 = x(x + 1)(x + 2)$.

Solution: If $x < -2$, then there is no solution. [1]
If $x = 0, -1, -2$, then $y = 0$. [1]
If $x \geq 1$, then $\gcd(x + 1, x(x + 2)) = 1$. Therefore $(x + 1)$ and $x(x + 2)$ are both perfect squares. But, $x(x + 2) = x^2 + 2x = (x + 1)^2 - 1$. This implies $(x + 1)^2$ and $(x + 1)^2 - 1$ are consecutive numbers which are squares. This is not possible. [4]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b - c)f(a) + (c - a)f(b) \leq (b - a)f(c)$.

Solution: Consider any $c \in (a, b)$.
Applying Lagrange's Mean Value Theorem to f on $[a, c]$ we get, there exists $x_1 \in (a, c)$ such that $\dfrac{f(c) - f(a)}{c - a} = f'(x_1)$. [2]
Applying Lagrange's Mean Value Theorem to f on $[c, b]$ we get, there exists $x_2 \in (c, b)$ such that $\dfrac{f(b) - f(c)}{b - c} = f'(x_2)$. [2]
Now f' is a decreasing function. Therefore $x_1 < \ x_2$ implies $f'(x_2) < \ f'(x_1)$.
Hence $\dfrac{f(b) - f(c)}{b - c} < \dfrac{f(c) - f(a)}{c - a}$. Therefore $f(b) - f(c)c - a < \ f(c) - f(a)b - c$.
This implies $(b - c)f(a) + (c - a)f(b) \leq (b - c + c - a)f(c) = (b - a)f(c)$. [2]

4. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial with integer coefficients such that a_0, a_3 and $f(1)$ are odd. Show that f has no rational root.

Solution: Suppose f has a rational root, say, $\dfrac{p}{q}$. Then $f(\dfrac{p}{q}) = 0$. Therefore $a_0q^3 + a_1pq^2 + a_2p^2q + a_3p^3 = 0$.
This implies $q|a_3$ and $p|a_0$. Since a_0, a_3 are odd, p, q are also odd. [3]
Also $a_0q^3 + a_1pq^2 + a_2p^2q + a_3p^3 = 0$ implies $a_0q^3 + a_1pq^2 + a_2p^2q + a_3p^3 \equiv 0 (mod 2)$.
Since p, q are odd, we have $a_0 + a_1 + a_2 + a_3 \equiv 0 (mod\ 2)$. This is contradiction because $f(1) = a_0 + a_1 + a_2 + a_3$ is odd. Hence f has no rational root. [3]

5. Let $f\ :\ \mathbb{R} \to \mathbb{R}$ be a function such that $f(x - y) = f(x)f(y)$ and $f(x) \neq 0$ for all x. Find $f(3)$.

Solution I: If $y = x$, then $f(0) = [f(x)]^2$. If, in particular, $x = 0$, then $f(0) = [f(0)]^2$. But, $f(x) \neq 0$. Therefore $f(0) = 1$. Now $[f(x)]^2 = 1$ implies $f(x) = \pm 1$. [2]
Put $x = 1, y = \dfrac{1}{2}$, Then $f(\dfrac{1}{2}) = f(1)\dfrac{1}{2}$. But, $f(x) \neq 0$. Therefore $f(1) = 1$. [2]
Also $f(2 - 1) = f(2)f(1) = f(1)$. Therefore $f(2) = 1$.
Also $f(3 - 2) = f(3)f(2) = f(1)$. Therefore $f(3) = 1$. [2]
OR
Solution II: Put $x = 3, y = 1.5$, then $f(1.5) = f(3 - 1.5) = f(3)f(1.5)$. Hence $f(3) = 1$. [6]

Part III

1. Prove that the equation $e^x - ln(x) - 2^{2014} = 0$ has exactly two positive real roots. [12]

Solution: Let $f(x) = e^x - \log x - 2^{2014},\ x > 0$. So $f'(x) = e^x - \frac{1}{x}$, and $f''(x) = e^x + \frac{1}{x^2}$. [2]
So $f'(1) = e - 1 > 0$. As $\lim_{x \to 0+} f'(x) = -\infty$, by the continuity of f', $f'(x) = 0$ for some value $x = a$. [2]
Since $f''(x) > 0$ for $x > 0$, f' strictly increases for $x > 0$. Therefore f has a unique critical point $x = a$. Note that f has minimum value at $x = a$ as $f''(a) > 0$. This minimum value is negative because, $f(a) < f(1) < 0$. [2]
As $\lim_{x \to 0+} \log x = -\infty$, it can be seen that $f(x)$ is positive when x is near 0. Also, as $\lim_{x \to \infty} e^{-x} \log x = 0$, it can be seen that $f(x)$ is positive when x is large. Thus there exist b, c with $0 < b < a$ and $a < c$ such that $f(b) > 0$, $f(a) < 0$ and $f(c) > 0$. Hence by continuity of f, $f(x) = 0$ has exactly two real roots : one root in each of the intervals (b, a) and (a, c). [6]
Note: Students may draw graphs for the proof. If only graph of $f(x)$ is drawn, give 4 marks. Further if there is more explanation with the graph, additional 4 marks may be given. If the argument is complete, all marks may be given.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-constant function satisfying $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that

   (a) $f(x) \neq 0$ for all $x \in \mathbb{R}$;

   (b) $f(x) > 0$ for all $x \in \mathbb{R}$;

   (c) If f is differentiable at 0, then f is differentiable on $\mathbb{R}$ and there exists some real number β such that $f(x) = \beta^x$ for all $x \in \mathbb{R}$. [12]

Solution: (a) Suppose for some real number x_0, we have $f(x_0) = 0$. Then for any $x \in \mathbb{R}$, $f(x) = f(x - x_0 + x_0) = f(x - x_0)f(x_0) = 0$. Therefore f is a constant function, which is a contradiction. Hence $f(x) \neq 0$ for all $x \in \mathbb{R}$ [2]

(b) Observe that $f(x) = f(\frac{x}{2})f(\frac{x}{2}) = [f(\frac{x}{2})]^2$.
Therefore $f(x) > 0$ for all $x \in \mathbb{R}$. [2]

(c) Suppose $f'(0)$ exists. Also given condition implies $f(0) = 1$.
Then for any $x \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h}$$
$$= \lim_{h \to 0} f(x)[\frac{f(h) - 1}{h}]$$
$$= \lim_{h \to 0} f(x)[\frac{f(h) - f(0)}{h}] = f(x)f'(0).$$

Therefore f is differentiable on $\mathbb{R}$ and $f'(x) = f(x)f'(0)$. [4]
This implies $\frac{f'}{f} = k = f'(0)$.
Integrating both sides with respect to x, we get,
$\log f(x) = kx + c$.
Now, $x = 0$ implies $\log f(0) = \log 1 = c = 0$.
Hence, $f(x) = e^{kx} = \beta^x$, where $\beta = e^k$. [4]

3. Let n be a natural number. Suppose $P_1, P_2, \cdots, P_n$ are points on a circle of radius 1. Prove that

$$\sum_{1\le i<j\le n} d(P_i, P_j)^2 \le n^2\,,$$

where for points X and Y in the plane, we denote by $d(X, Y)$ the distance between them. Prove that equality can hold for every natural number n. [13]

Solution: Consider a circle of radius 1 with center at origin. If $\overline{r_1}, \overline{r_2}, \cdots, \overline{r_n}$ are position vectors of $P_1, P_2, \cdots, P_n$ respectively, then we want to prove that $\sum(\overline{r_i} - \overline{r_j}) \cdot (\overline{r_i} - \overline{r_j}) \le 2n^2$. [3]

Now $\sum(\overline{r_i} - \overline{r_j}) \cdot (\overline{r_i} - \overline{r_j})$

$= (\overline{r_1} - \overline{r_2}) \cdot (\overline{r_1} - \overline{r_2}) + (\overline{r_1} - \overline{r_3}) \cdot (\overline{r_1} - \overline{r_3}) + \cdots + (\overline{r_1} - \overline{r_n}) \cdot (\overline{r_1} - \overline{r_n}) + (\overline{r_2} - \overline{r_1}) \cdot (\overline{r_2} - \overline{r_1}) + (\overline{r_2} - \overline{r_3}) \cdot (\overline{r_2} - \overline{r_3}) + \cdots + (\overline{r_2} - \overline{r_n}) \cdot (\overline{r_2} - \overline{r_n}) + \cdots + (\overline{r_n} - \overline{r_{n-1}}) \cdot (\overline{r_n} - \overline{r_{n-1}})$

$= 2(n-1)(\overline{r_1} \cdot \overline{r_1} + \overline{r_2} \cdot \overline{r_2} + \cdots + \overline{r_n} \cdot \overline{r_n}) - 2\sum_{i\ne j} 2\overline{r_i} \cdot \overline{r_j}$

$= 2(n-1)n - 2\sum_{i\ne j} 2\overline{r_i} \cdot \overline{r_j}$ [4]

$= 2(n-1)n + 2(\overline{r_1} \cdot \overline{r_1} + \overline{r_2} \cdot \overline{r_2} + \cdots + \overline{r_n} \cdot \overline{r_n}) - 2(\overline{r_1} \cdot \overline{r_1} + \overline{r_2} \cdot \overline{r_2} + \cdots + \overline{r_n} \cdot \overline{r_n}) - 2\sum_{i\ne j} 2\overline{r_i} \cdot \overline{r_j}$

$= 2n^2 - 2n + 2n - (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n}) \cdot (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n})$

$= 2n^2 - (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n}) \cdot (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n}) \le 2n^2$. [4]

Equality holds if $\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n} = \overline{0}$. [2]

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(0) = 0$. Suppose that $|f(z) - f(w)| = |z - w|$ for any $w \in \{0, 1, i\}$ and $z \in \mathbb{C}$. Prove that $f(z) = \alpha z$ or $f(z) = \alpha\overline{z}$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. [13]

Solution: Let $\alpha = f(1)$ and $\beta = f(i)$.
By the hypothesis, $|f(z)| = |z|, |f(z) - \alpha| = |z - 1|$ and $|f(z) - \beta| = |z - i|$ for all $z \in \mathbb{C}$.
In particular, by substituting $z = 1, i$ in the above equalities, we obtain
$|\alpha| = 1 = |\beta|,\ |\alpha - \beta| = \sqrt{2}$. [4]
We can write
$\alpha^2 + \beta^2$
$= \alpha^2|\beta|^2 + \beta^2|\alpha|^2$
$= \alpha^2\beta\overline{\beta} + \beta^2\alpha\overline{\alpha}$
$= \alpha\beta(\alpha\overline{\beta} + \beta\overline{\alpha})$
$= \alpha\beta(\alpha\overline{\alpha} + \beta\overline{\beta} - (\alpha - \beta)(\overline{\alpha} - \overline{\beta}))$
$= \alpha\beta(|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2)$
$= \alpha\beta(1 + 1 - 2) = 0,$
yielding $\beta = \epsilon\alpha$, where $\epsilon = \pm i$. [4]
Simplifying $|f(z) - \alpha|^2 = |z - 1|^2$, we get $\overline{\alpha}f(z) + \alpha\overline{f(z)} = z + \overline{z}$ and
simplifying $|f(z) - \beta|^2 = |z - i|^2$, we get $\overline{\alpha}f(z) - \alpha\overline{f(z)} = -\epsilon iz + \epsilon i\overline{z}$ for all $z \in \mathbb{C}$.
Adding up these equalities, we obtain

$$2\overline{\alpha}f(z) = (1 - \epsilon i)z + (1 + \epsilon i)\overline{z}.$$

If $\epsilon = i$, then $f(z) = \alpha z$ and if $\epsilon = -i$, then $f(z) = \alpha\overline{z}$ for all $z \in \mathbb{C}$. [5]