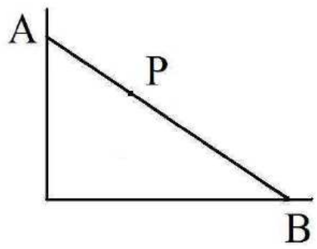
B.Stat. (Hons.) & B.Math. (Hons.) Admission Test: 2012

Multiple-Choice Test
Time: 2 hours

1. A rod AB of length 3 rests on a wall as follows:



P is a point on AB such that $AP : PB = 1 : 2$. If the rod slides along the wall, then the locus of P lies on

(A) $2x + y + xy = 2$

(B) $4x^2 + y^2 = 4$

(C) $4x^2 + xy + y^2 = 4$

(D) $x^2 + y^2 - x - 2y = 0$.

2. Consider the equation $x^2 + y^2 = 2007$. How many solutions (x, y) exist such that x and y are positive integers?

(A) None

(B) Exactly two

(C) More than two but finitely many

(D) Infinitely many.

3. Consider the functions $f_1(x) = x, f_2(x) = 2 + \log_e x,\ x > 0$ (where e is the base of natural logarithm). The graphs of the functions intersect

(A) once in $(0, 1)$ and never in $(1, \infty)$

(B) once in $(0, 1)$ and once in (e^2, ∞)

(C) once in $(0, 1)$ and once in (e, e^2)

(D) more than twice in $(0, \infty)$.

4. Consider the sequence

$$u_n = \sum_{r=1}^{n} \frac{r}{2^r}, n \geq 1.$$

Then the limit of u_n as $n \to \infty$ is

(A) 1 (B) 2 (C) e (D) 1/2.

5. Suppose that z is any complex number which is not equal to any of $\{3, 3\omega, 3\omega^2\}$ where ω is a complex cube root of unity. Then

$$\frac{1}{z-3} + \frac{1}{z-3\omega} + \frac{1}{z-3\omega^2}$$

equals

(A) $\frac{3z^2+3z}{(z-3)^3}$ (B) $\frac{3z^2+3\omega z}{z^3-27}$ (C) $\frac{3z^2}{z^3-3z^2+9z-27}$ (D) $\frac{3z^2}{z^3-27}$.

6. Consider all functions $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ which are one-one, onto and satisfy the following property:

if $f(k)$ is odd then $f(k+1)$ is even, $k = 1, 2, 3$.

The number of such functions is

(A) 4 (B) 8 (C) 12 (D) 16.

7. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then

(A) f is not continuous

(B) f is differentiable but f' is not continuous

(C) f is continuous but $f'(0)$ does not exist

(D) f is differentiable and f' is continuous.

8. The last digit of $9! + 3^{9966}$ is

(A) 3 (B) 9 (C) 7 (D) 1.

9. Consider the function

$$f(x) = \frac{2x^2+3x+1}{2x-1},\ 2 \leq x \leq 3.$$

Then

(A) maximum of f is attained inside the interval $(2,3)$

(B) minimum of f is $28/5$

(C) maximum of f is $28/5$

(D) f is a decreasing function in $(2,3)$.

10. A particle P moves in the plane in such a way that the angle between the two tangents drawn from P to the curve $y^2 = 4ax$ is always 90°. The locus of P is

(A) a parabola (B) a circle (C) an ellipse (D) a straight line.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = |x^2 - 1|,\ x \in \mathbb{R}.$$

Then

(A) f has a local minima at $x = \pm 1$ but no local maximum

(B) f has a local maximum at $x = 0$ but no local minima

(C) f has a local minima at $x = \pm 1$ and a local maximum at $x = 0$

(D) none of the above is true.

12. The number of triples (a, b, c) of positive integers satisfying

$$2^a - 5^b 7^c = 1$$

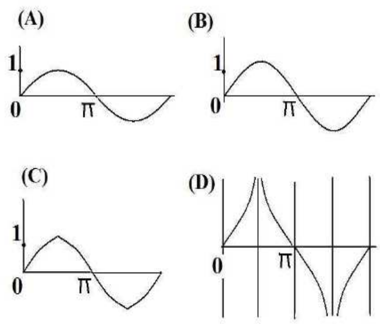
is

(A) infinite (B) 2 (C) 1 (D) 0.

13. Let a be a fixed real number greater than -1. The locus of $z \in \mathbb{C}$ satisfying $|z - ia| = Im(z) + 1$ is

(A) parabola (B) ellipse (C) hyperbola (D) not a conic.

14. Which of the following is closest to the graph of $\tan(\sin x), x > 0$?



15. Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}$ given by

$$f(x) = \frac{2x}{x-1}.$$

Then

(A) f is one-one but not onto

(B) f is onto but not one-one

(C) f is neither one-one nor onto

(D) f is both one-one and onto.

16. Consider a real valued continuous function f satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Let

$$g(t) = \int_0^t f(x)\,dx, \quad t \in \mathbb{R}.$$

Define $h(t) = \lim_{n\to\infty} \frac{g(t+n)}{n}$, provided the limit exists. Then

(A) $h(t)$ is defined only for $t = 0$

(B) $h(t)$ is defined only when t is an integer

(C) $h(t)$ is defined for all $t \in \mathbb{R}$ and is independent of t

(D) none of the above is true.

17. Consider the sequence $a_1 = 24^{1/3}$, $a_{n+1} = (a_n + 24)^{1/3}, n \geq 1$. Then the integer part of a_{100} equals

(A) 2 (B) 10 (C) 100 (D) 24.

18. Let $x, y \in (-2, 2)$ and $xy = -1$. Then the minimum value of

$$\frac{4}{4 - x^2} + \frac{9}{9 - y^2}$$

is

(A) 8/5 (B) 12/5 (C) 12/7 (D) 15/7.

19. What is the limit of

$$\left(1 + \frac{1}{n^2 + n}\right)^{n^2 + \sqrt{n}}$$

as $n \to \infty$?

(A) e (B) 1 (C) 0 (D) ∞.

20. Consider the function $f(x) = x^4 + x^2 + x - 1, x \in (-\infty, \infty)$. The function

(A) is zero at $x = -1$, but is increasing near $x = -1$

(B) has a zero in $(-\infty, -1)$

(C) has two zeros in $(-1, 0)$

(D) has exactly one local minimum in $(-1, 0)$.

21. Consider a sequence of 10 A's and 8 B's placed in a row. By a run we mean one or more letters of the same type placed side by side. Here is an arrangement of 10 A's and 8 B's which contains 4 runs of A and 4 runs of B:

$$A\,A\,A\,B\,B\,A\,B\,B\,B\,A\,A\,B\,A\,A\,A\,A\,B\,B$$

In how many ways can 10 A's and 8 B's be arranged in a row so that there are 4 runs of A and 4 runs of B?

(A) $2\binom{9}{3}\binom{7}{3}$ (B) $\binom{9}{3}\binom{7}{3}$ (C) $\binom{10}{4}\binom{8}{4}$ (D) $\binom{10}{5}\binom{8}{5}$.

22. Suppose $n \geq 2$ is a fixed positive integer and

$$f(x) = x^n|x|, x \in \mathbb{R}.$$

Then

(A) f is differentiable everywhere only when n is even

(B) f is differentiable everywhere except at 0 if n is odd

(C) f is differentiable everywhere

(D) none of the above is true.

23. The line $2x + 3y - k = 0$ with $k > 0$ cuts the x axis and y axis at points A and B respectively. Then the equation of the circle having AB as diameter is

(A) $x^2 + y^2 - \frac{k}{2}x - \frac{k}{3}y = k^2$

(B) $x^2 + y^2 - \frac{k}{3}x - \frac{k}{2}y = k^2$

(C) $x^2 + y^2 - \frac{k}{2}x - \frac{k}{3}y = 0$

(D) $x^2 + y^2 - \frac{k}{3}x - \frac{k}{2}y = 0$.

24. Let $\alpha > 0$ and consider the sequence

$$x_n = \frac{(\alpha+1)^n + (\alpha-1)^n}{(2\alpha)^n}, n = 1, 2, \ldots.$$

Then $\lim_{n\to\infty} x_n$ is

(A) 0 for any $\alpha > 0$

(B) 1 for any $\alpha > 0$

(C) 0 or 1 depending on what $\alpha > 0$ is

(D) 0, 1 or ∞ depending on what $\alpha > 0$ is.

25. If $0 < \theta < \pi/2$ then

(A) $\theta < \sin\theta$

(B) $\cos(\sin\theta) < \cos\theta$

(C) $\sin(\cos\theta) < \cos(\sin\theta)$

(D) $\cos\theta < \sin(\cos\theta)$.

26. Consider a cardboard box in the shape of a prism as shown below. The length of the prism is 5. The two triangular faces ABC and $A'B'C'$ are congruent and isosceles with side lengths 2,2,3. The shortest distance between B and A' along the surface of the prism is

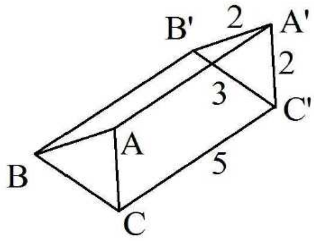
(A) $\sqrt{29}$ (B) $\sqrt{28}$ (C) $\sqrt{29-\sqrt{5}}$ (D) $\sqrt{29-\sqrt{3}}$

27. Assume the following inequalities for positive integer k:

$$\frac{1}{2\sqrt{k+1}} < \sqrt{k+1} - \sqrt{k} < \frac{1}{2\sqrt{k}}.$$

The integer part of

$$\sum_{k=2}^{9999} \frac{1}{\sqrt{k}}$$



equals

(A) 198 (B) 197 (C) 196 (D) 195.

28. Consider the sets defined by the inequalities

$$A = \{(x, y) \in \mathbb{R}^2 : x^4 + y^2 \leq 1\},\ B = \{(x, y) \in \mathbb{R}^2 : x^6 + y^4 \leq 1\}.$$

Then

(A) $B \subseteq A$

(B) $A \subseteq B$

(C) each of the sets $A - B, B - A$ and $A \cap B$ is non-empty

(D) none of the above is true.

29. The number

$$\left(\frac{2^{10}}{11}\right)^{11}$$

is

(A) strictly larger than $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2\binom{10}{5}$

(B) strictly larger than $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2$ but strictly smaller than $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2\binom{10}{5}$

(C) less than or equal to $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2$

(D) equal to $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2\binom{10}{5}$.

30. If the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ are in geometric progression then

(A) $b^2 = ac$ (B) $a^2 = b$ (C) $a^2b^2 = c^2$ (D) $c^2 = a^2d$.

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Short-Answer Type Test
Time: 2 hours

1. Let X, Y, Z be the angles of a triangle.

(i) Prove that

$$\tan\frac{X}{2}\tan\frac{Y}{2}+\tan\frac{X}{2}\tan\frac{Z}{2}+\tan\frac{Z}{2}\tan\frac{Y}{2}=1.$$

(ii) Using (i) or otherwise prove that

$$\tan\frac{X}{2}\tan\frac{Y}{2}\tan\frac{Z}{2}\leq\frac{1}{3\sqrt{3}}.$$

2. Let α be s real number. Consider the function

$$g(x)=(\alpha+\mid x\mid)^2e^{(5-|x|)^2},\ -\infty<x<\infty\cdot\infty.$$

(i) Determine the values of α for which g is continuous at all x.

(ii) Determine the values of α for which g is differentiable at all x.

3. Write the set of all positive integers in triangular array as

1	3	6	10	15	·	·
2	5	9	14	·	·	·
4	8	13	·	·	·	·
7	12	·	·	·	·	·
11	·	·	·	·	·	·

Find the row number and column number where 20096 occurs. For example 8 appears in the third row and second column.

4. Show that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real root.

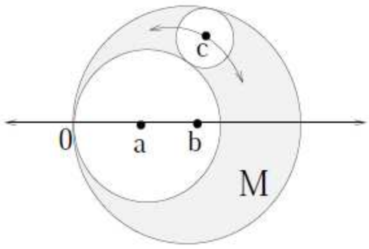
5. Let m be a natural number with digits consisting entirely of 6's and 0's. Prove that m is not the square of a natural number.

6. Let $0 < a < b$.

(i) Show that amongst the triangles with base a and perimeter $a + b$ the maximum area is obtained when the other two sides have equal length $\frac{b}{2}$.

(ii) Using the result (i) or otherwise show that amongst the quadrilateral of given perimeter the square has maximum area.

7. Let $0 < a < b$. Consider two circles with radii a and b and centers $(a, 0)$ and $(0, b)$ respectively with $0 < a < b$. Let c be the center of any circle in the crescent shaped region M between the two circles and tangent to both (See figure below). Determine the locus of c as its circle traverses through region M maintaining tangency.



8. Let $n \geq 1$, and $S = \{1, 2, \ldots, n\}$.For a function $f : S \to S$, a subset $D \subset S$ is said t be invariant under f, if $f(x) \in D$ for all $x \in D$. Note that the emptyset and S are invariant for all f. Let $\deg(f)$ be the number of subsets of S invariant under f.

(i) Show that there is a function $f : S \to S$ such that $\deg(f) = 2$.

(ii) Further show that for any k such that $1 \leq k \leq n$ there is a function $f : S \to S$ such that $\deg(f) = 2^k$.